METHOD FOR PRODUCING ACCURATE VALUES FOR ARCTAN

It was over a decade ago that I first stumbled upon an approximation technique for the tan(x) and arctan(x) functions based upon the use of even Legendre Polynomials of high order. The method, which is much simpler than other existing approximation techniques, has now received considerable attention in the literature and is known as the KTL Method. Since that time I have extended the technique to several other functions including the Gaussian, ln(1+x), and exp(a). See our most recent article found at-

http://www2.mae.ufl.edu/~uhk/KTL-METHOD.pdf

We want in the present note to look at an improved approximation for arctan(1/a) which will be valid over the full range o<a<∞. Unlike in an earlier note-

http://www2.mae.ufl.edu/~uhk/ARCTAN-NOTE.pdf

,which I worked on in conjunction with Sidey Timmins, the present approach will involve no extrapolations and instead presents values to any desired order of accuracy at any angle θ=a180/π. Also we will no longer be limited to lower values of n in the Legendre polynomials. A single formula which predicts values for arctan(1/a) with 20 digits accuracy and up will be presented and evaluated for several test cases.

Our starting point in the present analysis is the integral-

$$J(a,n)=\int_{x=0}^{1}\frac{P(2n,x)}{(x^2+a^2)}dx$$

, where P(2n,x) are the even Legendre polynomials of order 2n. On integrating J(a,n) we find-

$$J(a,n)=N(a,n)\arctan(1/a)+M(a,n)$$

, where N (a,n(and M(a,n) are polynomials in 'a' which increase in size as n is increased. One notices that J(a,n) approaches zero as n>>1. As a result we have the approximation-

$$\arctan(1/a)\approx Q(a,n)=-\frac{M(a,n)}{N(a,n)}$$

This approximation will get better the larger n becomes. It requires that a>1 in order to guarantee that the variation of the denominator portion of the integrand varies only slowly over the range 0<x<1. When a<1 one can use the trigonometric identity

$$\arctan(a)+\arctan(1/a)=\pi/2$$

which, when used in conjunction with appropriate Heaviside functions, allows us to express the quotient function as-

$$Q(a,n)=\left\{\frac{\pi}{2}+\frac{M[(1/a),n]}{N[(1/a),n)}\right\}[Heaviside(a-0)-Heaviside(a-1)]$$
$$+\left\{\frac{-M(a,n)}{N(a,n)}\right\}[Heaviside(a-1)-Heaviside(a-b)] \quad with \quad b>2$$

for the full range 0<a<∞.

The weakest part of the Q(a,n) approximation is expected for a=1 radian (57.2957deg) . If we can make this point accurate to N decimal places then the approximation for arctan(1/a) for any other ‘a’ will be better.

To demonstrate how the KTL Method allows one to get an approximation for arctan(1/a) for the full range 0<a<∞, we will start with the low value of n=2. Here the Quotient reads-

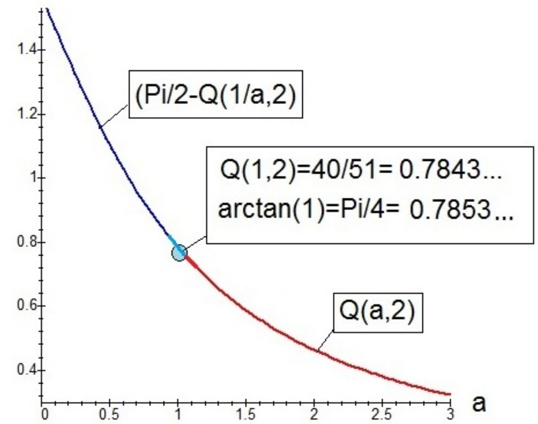
$$Q(a,2)=\frac{105a^3+55a}{105a^4+90a^2+9} \quad and \quad Q(1/a,4)=\frac{105a+55a^3}{105+90a^2+9a^4}$$

Note to go from one Q to the other we only need to exchange the powers of ‘a’.
It leads us to the conclusion that-

arctan(1/a)≈Q(a,4) when 1<a<b and arctan(1/a)≈(π/2-Q(1/a,4)] when 0<a<1

A plot of this two part function follows-

APPROXIMATION FOR ARCTAN(1/a) FOR n=2



$$Q(a,2)=\frac{105a^3+55a}{105a^4+90a^2+9}$$

Note the continuity of the two functions at a=1. Even more accurate results will follow as n is increased beyond n=2.

We next want to see how far our arctan(1/a) approximation falls from the exact value. This is quickest to accomplish by evaluating [π/4-Q(1,n)] for various values of n. These calculations are easy to carry out on my PC and lead to the following table-

n	arctan(1)-Q(1,n)
2	0.108443×10^{-2}
4	0.957124×10^{-6}
6	0.834887×10^{-9}
8	0.725947×10^{-12}
10	0.630382×10^{-15}
12	0.547027×10^{-18}
14	0.474507×10^{-21}

Since we expect the weakest part of the arctan approximation to occur at a=1, it is clear from the table that n=14 will guarantee an accuracy of better than 20 digits for all values of ‘a’. The Quotient Q(a,14) which offers this degree of accuracy will have N(a,14) be a 28 degree polynomial and M(a,14) a 27 degree polynomial. Fortunately we don’t have to write these long polynomials down on paper since they can be stored in the computer memory.

Here is a table of the approximations for arctan(1/a) using Q(a,14) evaluated at a=1/4,1/2,1, 2, and 4-

a	Arctan(1/a) approximation using Q(a,14)
4	0.24497866312686415417208248121127581091414409838118…
2	0.463647609000806116214256231461214…
1	0.785398163397448309615…
1/2	1.107148717794090503017065460178537…
1/4	1.32581766366803246505923921042847563118444060130636…

The agreement is exact to the points where … first appear. That is, from an accuracy of 21 digits at a=1 to an accuracy of 51 places at a=4 and ¼. One can always increase the accuracy of our Q(a,n) approximations by making n still larger. However, for most practical purposes twenty digit accuracy for all ‘a’ values for the arctan(1/a) using n=14 is more than sufficient. Note that no extrapolations are necessary to get correct arctan(1/a) values for any ‘a’ . One needs only to specify ‘a’ and n , the rest follows by a split-second computer evaluation of the quotient of two large polynomials in ‘a’.

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