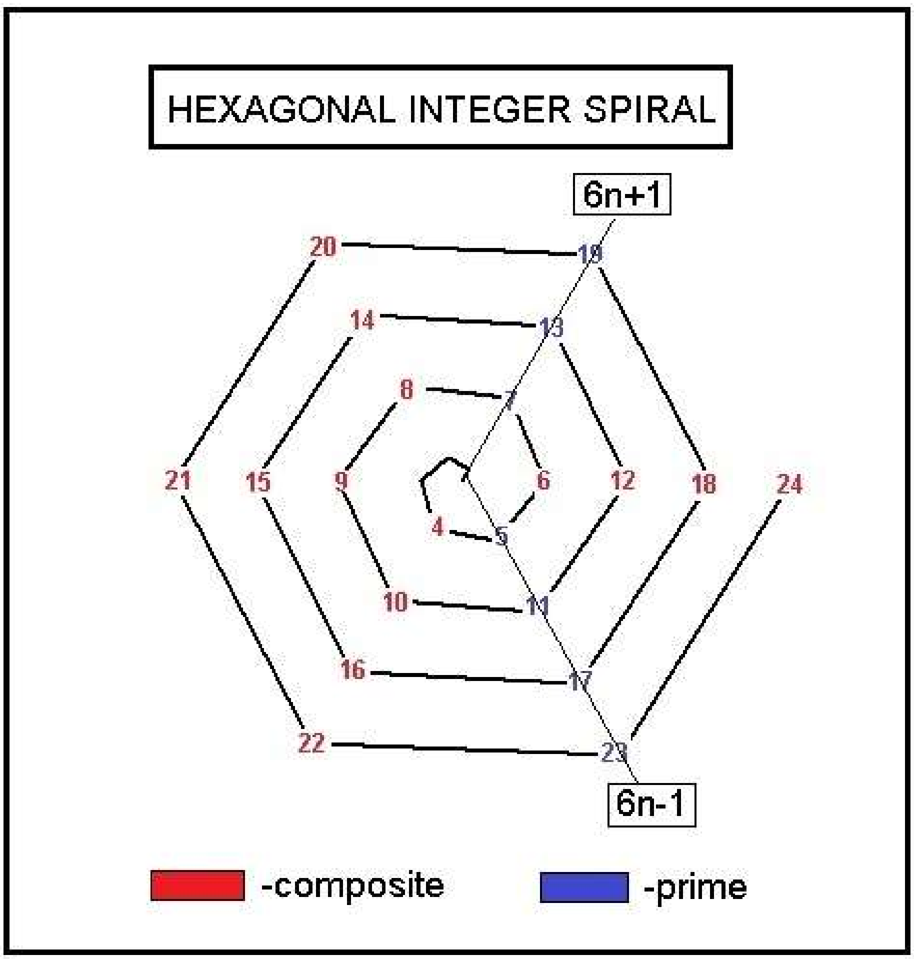
GEOMETRIC REPRESENTATION OF PRIME NUMBERS USING A HEXAGONA|L INTEGER SPIRAL

About a decade ago we came up with a new way to represent all primes five or greater by means of an integral spiral and the intersection of two radial lines 6n+1 and 6n-1 (https://mae.ufl.edu/~uhk/MORPHING-ULAM.pdf). A picture of this hexagonal spiral follows-



By counting the integers one step at a time in the counterclockwise sense, on sees that primes are found only along just two radial lines –

$$6n+1=7, 13, 19, 31,\ldots \quad \text{and} \quad 6n-1=5, 11, 17, 23$$

Since there are also some composite numbers falling along these two radial lines (for instance N=25), one can only make the restricted statement that-

A necessary but not sufficient condition is that all primes greater than three must have the form $6n\pm1$

We have found no exception to this rule, having tested numbers up to 100 digit length. Notice that any of the six possible radial lines intersecting the vertexes of the spiral contain integers differing by factors of six from each other. Thus in number theory language-

$$(6n+1) \bmod(6)=1 \qquad \text{and} \qquad (6n-1)\bmod(6)= 5(\text{or}-1)$$

We want in this note to discuss some of the other important properties of the integral spiral and show its length, role in determining twin primes, and discuss its connection with semi-primes.

Let us start with the location of the vertexes of the integer spiral. These are found in polar coordinates at-

$$[r,\theta]=[N, N\pi/3] \quad \text{with } N=0,1,2,3,4..$$

Using the following one line computer program -

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listplot([seq([N,N*Pi/3],N=0..24)],coords=polar,scaling=constrained,numpoints=3000,axes=none,thickness=2);
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produces the spiral pattern shown above. To get the distance between neighboring integers one uses the law of cosines. This produces-

$$L_N=sqrt(N^2+N+1)$$

The first turn around the integer spiral starting with N=0 and going to N=5 is equal in total length to-

S_1=sum(sqrt(N^2+N+1),N=0..5)=sqrt(1)+sqrt(3)+sqrt(7)+sqrt(13)+sqrt(21)+sqrt(31)=19.13369..

The total spiral length from N=0 up to 10th turn is-

S_{59}=evalf(sum(sqrt(N^2+N+1),N=0..59),10)=1801.9957..

Note that the spiral side lengths increase very rapidly with increasing N.

An interesting new result brought about by the integer spiral notation is the property of twin primes. These primes differ from each other by two units and hence can only have the form-

$$p=6n+1 \quad \text{and } q=6n-1$$

Adding them together means that p+q=12n. So one can quickly construct a twin prime table [ ] as follows-

[5,7]=12 , [11,13]=24, [17,19]=36, [29,31]=60], [41,43]=84 , [59,61]=120, [71,73]=144, [101,103]=204. [107,109]=216

Note that the term outside the square brackets always divides by 12 as expected. Although it has not yet been completely proven, there are expected to be an infinite numbers of such twin primes.

Another new result which can be generated from our integer spiral is the location of semi-primes. These numbers N=pq represent the product of two primes. For primes greater than three, these semi-primes must all lie along the radial lines 6n±1 as do the primes. Hence it must be true that N mod(6) equals 1 or 5 and so lies along the radial line 6n+1 and 6n-1. Let us demonstrate things for N=77. Here 77 mod(6)=-1 so it lies along radial line 6n-1. The primes p and q must then lie along 6n+1 and 6m-1, respectively. So one can write-

$$(6n+1)(6m-1)=6(13)-1$$

This is equivalent to-

$$m=(n+3)/(n+1)$$

so that the two prime components of 77 are 7 and 11. This procedure continuous to hold for any semi-prime although the solution of the algegbraic quotient becomes difficult to evaluate when N gets large, such as encountered in public key cryptography.

The simplest way to generate semi-primes is to reverse the problem and start with two primes. Take for example the primes p=599 and q=761. Here the semi-prime becomes N=455839 and a mod(6) operation yields 1. Hence this N lies along the 6n+1 radial line. Hence one has either –

(6n-1)(6m-1)=6(75973)+1 or (6n+1)(6m+1)=6(75973)+1

This produces the algebraic quotients-

m=(75973+n)/(6n-1) or m=(75973-n)/(6n+1)

After a lot of manipulation, the first of these quotients is satisfied by n=100 and m=127. This produces p=599 and q=761. We have recently discovered an alternate way to evaluate the components p and q of a semi-prime. It reads-

x^2-2x{sqrt(N)+n}+N=0

, where x=[p,q] and sqrt(N) represents the nearest integer of this root. This solution works particularly well when n is small as it will be if 2sqrt(N)$\approx (p+q)$. For n=455839 this is the case since 2(675)=1350≈ 1360. With just five trials (n=5), this quadratic for N=455839 solves as x=[599,761] . For most cases N will be associated with large ns and hence require numerous trials.

As a final observation concerning the hexagonal spiral let us read off the first twenty five or so primes lying along each of the two radial lines. The program for doing so is-

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for n from 1 to 40 do({n,6*n+1, isprime(6*n+1)})od;
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and-

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for n from 1 to 40 do({n,6*n-1, isprime(6*n-1)})od;
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Here are the resulting primes-

6n+1= 7, 13, 19, 31, 37, 43, 61, 67, 73, 79, 97, 103, 109, 127, 139, 151, 157, 163, 181,193, 199, 211, 223, 229, 241,..

and-

6n-1= 5, 11, 17, 23, 29, 41, 47, 53, 59, 71, 83, 89, 101, 107, 113, 131, 137, 149, 167, 173, 179 ,191, 197, 227, 233, 239,..

Notice that any two primes along either radial line 6n+1 or 6n-1 differ from each other by products of six. Thus 179-71=6(18) and 199-19=6(30). There are a few larger gaps found along these two radial lines. Two examples of gaps occur between 79 and 97 and between 113 and 131. The numbers in such gaps are either semi-primes or highr. Thus 85=5(17) and 91=7(13). Semi-primes typically have values of the number fraction-

$$f=[\mathrm{sigma}(N)-N-1]/N$$

equal less than one and just slightly above zero. So-

$$f(91)=(7+13)/[(7)(13)]= 20/91=0.219..$$

Note that for any semi-prime, containing only primes above three, we have the interesting general number fraction expression-

$$f\,(pq)=(p+q)/(pq) = [\sigma(pq)-pq-1]\,/(pg)$$

Here σ(pq) is the sigma function.

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