THE FUNCTION f(x)=1/(1-exp(x)) AND A NEW TYPE OF PASCAL TRIANGLE

About seven years ago while playing around with the function f(x)=1/(1-exp(x)) (see https://www2/mae.ufl.edu/~uhk/MORE-PASCAL.pdf) we first noticed that this odd function with a singularity at x=0 has the interesting property that-

f’=exp(x)/(1-exp(x))^2[1]

f”=exp(x)/(1-exp(x))^3[exp(x)+1]

f‴=exp(x)/(1-exp(x))^4[exp(2x)+4exp(x)+1]

f^{iV}=exp(x)/(1-exp(x))^5[exp(3x)+11exp(2x)+11exp(x)+1]

By bringing the terms not inside the square bracket on the right hand side to the left, we have the new two parameter function-

$$K(n,x)=\left(\frac{(1-\mathrm{e}\quad(x))^{n+1}}{\exp(x)}\right)\frac{d^n}{dx^n}\left\{\frac{1}{1-\exp(x)}\right\}$$

Here K(n,x) is a finite length series in exp(ax), with a=0,1,2,…(n-1). After setting x=0, one obtains the heretofore unknown Pascal-like triangle shown-

1

1 1

1 4 1

1 11 11 1

1 26 66 26 1

1 57 302 302 57 1

1 120 1191 2416 1191 120 1

1 247 4293 15619 15619 4293 247 1

An inspection of this triangle shows the interesting property that the sum of the elements in any given row n add up to precisely n!. Thus row seven adds up to 7!=5040. After some lengthy inspection we also find that any element in row n and column m designated by D[n,m] is given by the formula-

$$D[n,m]=(n+1-m)D[n-1,m-1]+mD[n-1,m]$$

Thus D[7,4]=4D[6,3]+4D[6,4]=4(302)+4(302)=2416 . Thus knowing the value of D[n-1,m] in the (n-1) row allows one to find any D[n,m] in the nth row. The symmetry of the triangle is also noted to occur about column m/2. Similar to the values of D[n,m] for a standard Pascal Triangle , the value of the elements D[n,m] approach that of a Gaussian when fixed n gets large. Note that in terms of the function K(n,x], we have that-

$$n!=\lim_{x\to 0} K(n,x)$$

This can be considered as a derivative representation for the integral $n! = \int_0^\infty (t^n) exp(-t)dt$.

The first six functional forms of K(n,x) for n=1,2,3,4,5,6 have the form-

K(1,x)=1

K(2,x)=exp(x)+1

K(3,x)=exp(2x)+4exp(x)+1

K(4,x)-exp(3x)+11exp(2x)+11exp(x)+1

K(5,x)=exp(4x)+26exp(3x)+66exp(2x)+26exp(x)+1

K(6,x)=exp(5x)+57exp(4x)+302exp(3x)+302exp(2x)+57exp(x)+1

The next K(n,x) can be read directly off of the above modified Pascal Triangle. It reads-

K(7,x)=exp(6x)+120exp(5x)+1191exp(4x)+2416exp(3x)+1191exp(2x)+120exp(x)+1

We note that the number of terms in each of these expanded forms for K(n,x) just equals n. Also the functional form of K(n,x) for fixed n increases monotonically with increasing x at a rate greater than exp((n-1)x).

An interesting side-light to K(n,x) occurs when one replaces exp(x) by x and drops the term x in the denominator. This produces the identity-

$$T(n)=(1-x)^{\wedge}(n+1)\frac{d^{\wedge}n}{dx^{\wedge}n}\{1/(1-x)\}=n!$$

Thus T(10)=3628800=10! . We also have the identity-

$$\int_0^\infty x^n \exp(-x)\, dx = (1\text{-}x)\text{\^{}}(n+1)\frac{d\text{\^{}}n}{dx\text{\^{}}n}\{1/(1\text{-}x)\}$$

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February 24, 2020
Gainesville, Florida