ALL ABOUT THE FIBONACCI SEQUENCE

One of the best known and most studied integer sequences is the Fibonacci Sequence-

f(n)={0,1,1,2,3,5,8,13,21,..}

It satisfies the functional equation-

f(n+2)=f(n+1)+f(n) subject to f(0)=0 and f(1)=1

That is, the nth element f(n) always equals the sum of the preceding two elements f(n-1) and f(n-2). The integer elements head toward infinity as n goes to infinity. The ratio of the elements R(n)=f(n+1)/f(n) approaches the value of the Golden Ratio-

φ =[1+sqrt(5)]/2=1.61803398874989484820458683437…

as indicated in the following table-

n	1	2	3	4	5	6	7	8	9	10	11	12
R(n)	1	2	3/2	5/3	8/5	13/8	21/13	34/21	55/34	89/55	144/89	233/144

We see that R(12)=1.618055 already lies very close to the value of φ. It is the purpose of this note to look at some of the additional properties of the Fibonacci Sequence.

Let us begin with the Fibonacci Sequence definition-

r^(n+2))=r^(n-1)+r^n

and solve for r when n=0. This yields-

r=[1$\pm$sqrt(5)]=[φ or $(\frac{1}{\varphi})$] .

From this last result we have-

f(n)=c1 $(\varphi^n) + c2(\frac{1}{\varphi})$^$n$

with the constants c1 and c2 determined from the initial conditions f(0)=0 and f(1)=1. That is-

0=c1+c2 and 1=c1 φ + c2/φ

These solve as c1=1/sqrt(5) and c2=-1/sqrt(5) . Thus we arrive at the famous Binet formula-

f(n)=[1/sqrt(5)]{[(1+sqrt(5))/2]^n-[1-sqrt(5))/2]^n}

giving us the value of any element in the Fibonacci Sequence. We find, for example, that-

f(5)=5, f(10)=55, f(20)=6765, and f(40)= 102334155

These numbers can also be obtained somewhat faster by the one line computer program-

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f[0]:=0;   f[1]:=1;  f[2]:=1  for n from 1 to 38 do f[n+2]:=f[n+1]+f[n] od;
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which yields all f[n]s from n=1 through 40 in a split second.

One can use the Binet Formula to quickly show that the ratio R(n)=f(n+1)/f(n) approaches the Golden Ratio as n goes to infinity. Specifically we have-

$$\mathrm{R(n)}=\varphi\{\frac{1-\frac{1}{\varphi^{2(n+1)}}}{1-\frac{1}{\varphi^{2(n)}}}\}$$

Since φ>1, the quotient within the curly bracket approaches one as n gets large. Thus we have-

$$\mathrm{R}(\infty)=\varphi=1{,}61803\ldots$$

The famous German artist and amateur mathematician Albrecht Duerer(1471-1528) was already aware of this result five hundred years ago. Fibonacci (alias Leonardo of Pisa) came up with his sequence in 1202. At that time Fibonacci also introduced the Arabic-Indian number system to Europe. This is the base ten number system, including zero, which we presently use and one far superior to the Roman system.

We point out that the Fibonacci Sequence can easily be altered by changing the starting values from f(1)=1 and f(2)=1 to f(1)=1 but f(2)=3. This approach produces the Lucas Number Sequence-

L(n)={1, 3, 4, 7, 11, 18, 29, 47, 76, 123,...)

For large n the ratio f(n+1)/f(n) approaches the Golden Ratio.This implies that the starting conditions f(1) and f(2) will generally not alter the final ratio result R(n)=f(n+1)/ f(n).

Another variation of the Fibonacci Sequence can be gotten via the functional expression-

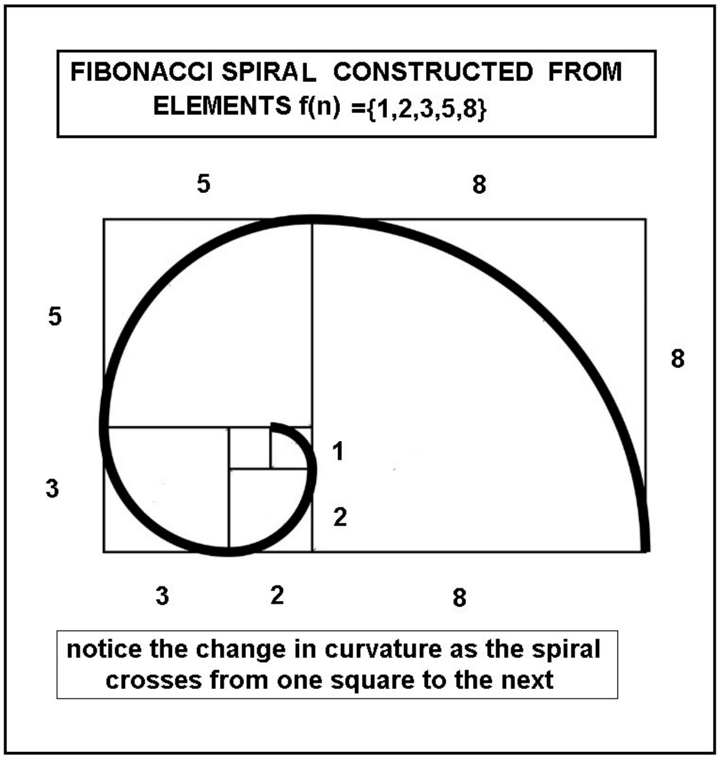
g(n+2)=g(n+1)-g(n) subject to g(1)=1 and g(2)=1

On solving, this last equality , we arrive at the sequence-

S={1, 1,0,-1,-1,0,1,1,0,-1,-1,0,…}

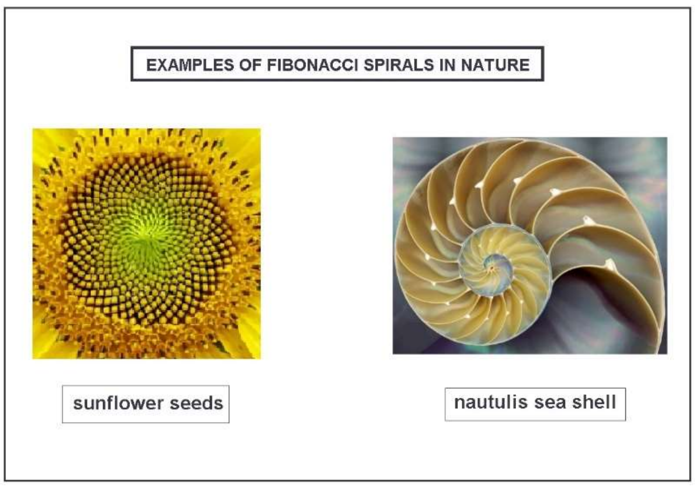
Note that all elements in this sequence are bounded between -1 and +1.

An interesting geometric figure can be constructed using the Fibonacci elements one through eight. This figure has the form of the following spiral-



The construction starts with a square of sidelength 8. In this square one places the quarter circle shown. This circle has a constant radius of radius of curvature of 8. Next a smaller square of sides 5 is drawn tangent to the original square. A quarter circle of constant radius of curvature r=5 is drawn within it. This represents the next part of the Fibonacci Spiral. Note the jump in curvature occurring where the spiral goes from the 8x8 square to the smaller 5 x 5 square . Next one draws an even smaller square of dimension 3 x 3. In it we place a quarter circle of radius r=3. This is followed by a still smaller square 2 x 2 containing a quarter circle of radius 2. The drawing is completed by adding a final unit square and its radius one circle. Adding all the quarter circles together produces the final Fibonacci Spiral designated by the heavy black curve. Fibonacci's Spiral is well known to the general public as is attested to by the following postage stamp-

Also the spiral resembles the structure of sunflower seeds and the cut-open skeleton of a nautilus sea shell.



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