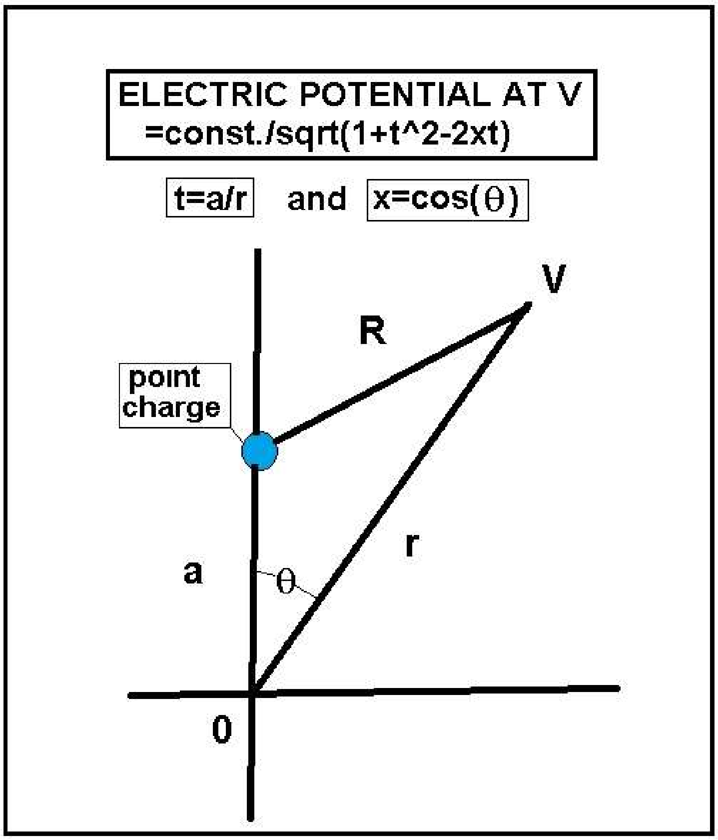
ORIGIN AND PROPERTIES OF LEGENDRE POLYNOMIALS

Back in 1782 the famous French mathematician A.M.Legendre came up with a new set of finite length polynomials now known as Legendre Polynomials. Their origin is the electric potential measured at point V at distance R from a single electric charge as shown in the following picture-



The potential at V from this point charge goes as const/R. Using the Law of Cosines and setting any common const to unity, the potential at V is-

$$1/\mathrm{sqrt}(1+t^2-2xt)$$

, with the two new variables t=a/R and x=cos(θ). On carrying out a series expansion in t, one finds-

$$1/\mathrm{sqrt}(1+t^2-2xt)=1+[x]t+[(3x^2-1)/2]\ t^2+[(5x^3-3x)/2]\ t^3+ O(t^4))$$

The terms in the square brackets are the Legendre Polynomials. The first few read-

$P(0,x)=1$, $P(1,x)=x$, $P(2,x)=(3x^2-1)/2$, and $P(3,x)=5x^3-3x)/2$

Generalizing, we come up with the basic identity for Legendre Polynomials-

1/sqrt(1+t^2-2xt)=$\sum_{n=0}^{\infty} P(n,x)t\text{^}n$

Next we differentiate this last equality with respect to t to get –

(x-t)$\sum_{n=0}^{\infty} P(n,x)t^n$=(1+t^2-2xt)$\sum_{n=0}^{\infty} nP(n,x)t\text{^}(n-1)$.

On expanding, this yields-

x$\sum_{n=0}^{\infty} P(n,x)t^n - \sum_{n=0}^{\infty} P(n,x)t\text{^}(n+1)$=

$\sum_{n=0}^{\infty} nP(n,x)t\text{^}(n-1)$ +$\sum_{n=0}^{\infty} nP(n,x)t^{n+1} - 2x\sum_{n=0}^{\infty} nP(n,x)t\text{^}n$

Converting the above five sums to the same powers t^n, these produce the Bonnet recurrence relation-

(n+1)P(n+1,x)=(2n+1)x P(n,x)-nP(n-1,x)

By setting n=2, we find, for example, that –

P(3,x)=[5xP(2,x)-2P(1,x)]/3=(5x^3-3x)/2

Starting with P(0,x)=1 and P(1,x)=x this recurrence relation can be used to quickly find all positice integer Legendre Polynomials. Here is a jpg for the first ten-

FIRST TEN LEGENDRE POLYNOMIALS USING BONNET'S RECURRENCE RELATION

$$P_0 := 1$$

$$P_1 := x$$

$$P_2 := -\frac{1}{2}+\frac{3}{2}x^2$$

$$P_3 := -\frac{3}{2}x+\frac{5}{2}x^3$$

$$P_4 := \frac{3}{8}-\frac{15}{4}x^2+\frac{35}{8}x^4$$

$$P_5 := \frac{15}{8}x-\frac{35}{4}x^3+\frac{63}{8}x^5$$

$$P_6 := -\frac{5}{16}+\frac{105}{16}x^2-\frac{315}{16}x^4+\frac{231}{16}x^6$$

$$P_7 := -\frac{35}{16}x+\frac{315}{16}x^3-\frac{693}{16}x^5+\frac{429}{16}x^7$$

$$P_8 := \frac{35}{128}-\frac{315}{32}x^2+\frac{3465}{64}x^4-\frac{3003}{32}x^6+\frac{6435}{128}x^8$$

$$P_9 := \frac{315}{128}x-\frac{1155}{32}x^3+\frac{9009}{64}x^5-\frac{6435}{32}x^7+\frac{12155}{128}x^9$$

Note that these polynomials have finite length with the highest powers of x equal to n. They are either even or odd functions with n zeros in -1<x<+1. One also has the integral values-

$$\int_{x=-1}^{1} P(n,x)P(m,x)dx = [2/(2n+1)]\,\delta nm$$

, where the delta represents the Kronecker delta. This orthogonality condition reduces to –

$$\int_{x=-1}^{1} P(n,x)d\,x = 0$$

on setting m=0. Both relations follow directly from the above list of Legendre Polynomials.

Another way to generate P(n,x) is via the second order ODE-

(1-x^2)y”-2xy’+n(n+1)y=0

, where y=P(n,x). The simplest way to verify that this differential equation indeed satisfies the Legendre Polynomials is to assume it to be correct and then evaluate it starting with n=1. Here is what one finds-

n=1 produces (1-x^2)0-2x+2x=0

n=2 produces (1-x^2)3-2x(3x)+3(3x^2-1)=0

n=3 produces (1-x^2)15x-x(15x^2-3)+6(5x^3-3x)=0

So clearly the above differential equation has y=P(n,x) as one of its solutions.

A final alternate way to generate Legendre Polynomials is by the Rodrigues Formula-

P(n,x)=[1/(n!2^n)]nth derivative of[(x^2-1)^n]

To prove its validity one again writes out the first few terms to verify that it indeed yields P(n,x). We have for-

n=1 that (1/2)(2x)=x=P(1,x)

n=2 that (1/8)(12x^2-4)=(3x^2-1/2=P(2,x)

n=3 that (1/48)(d^3/dx^3[x^2-1)^3])=(5x^3-3x)/2 =P(3,x)

There are an infinite number of definite integrals involving Legendre Polynomials. Among these are the following-

$$\int_{x=0}^{1} \frac{P(n,x)}{a^2+x^2}\,dx \qquad \int_{x=0}^{1} P(n,x)\sin(ax)\,dx \qquad \text{and} \qquad \int_{x=0}^{1} P(n,x)\cosh(ax)\,dx$$

These are used as starting points in the KTL approximation method for finding highly accurate approximations for arctan(1/a) , tan(a) and tanh(a), respectively.

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February 19, 2024
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