USE OF MATHEMATICAL INDUCTION

A powerful method to sum certain series and to determine the values of certain integrals involves mathematical induction. We want to use this article to demonstrate the technique.

Let us begin with the infinite series of odd integers-

S={1+3+5+….+(2n-1)}

We see that this series has zero second differences meaning that it should equal the quadratic A+Bn+Cn^2. Equating the first few terms one finds-

1=A+B+C, 4=A+2B+4C, and 9=A+3B+9C

These solve as A=B=0 and C=1 to produce the result-

!+3+5+ +(2n-1) $= n\hat{}2$

So the first ten odd integers add up to 100.

Consider next the series=

S={1+4+9+16+ …+(n^2)}

Here the third difference in the series equals zero. Hence a summation of this series must have the form A+Bn+Cn^2+Dn^3 . Here one finds that-

A+B+C+D=1 A+2B+4C+8D=4 A+3B+9C+27D=14 and A+4B+16D+64=30

Solving we find A=0, B=1/6, C=1/2 and D=1/3. This produces the result-

S={1+4+9+16+ …+(n^2)=(1/3)n^3+(1/2)n^2+(1/6)n

For n=5 we get S=55.

Mathematical induction also works for certain integrals. One of the better known of these integrals is-

$$\int_{t=0}^{\infty} x\hat{}n\ exp(-x)dx = n! = \Gamma(n+1)$$

Using mathematical induction, one finds , via integration by parts, that-

$$\int_{x=0}^{\infty} xexp(-x)dx = 1$$

$$\int_{x=0}^{\infty} x^2 exp(-x)dx = 2$$

$$\int_{x=0}^{\infty} x^3 exp(-x)dx = 6$$

$$\int_{x=0}^{\infty} x^4 exp(-x)dx = 24$$

One recognizes the sequence 1, 2, 6, 24 to just be 1!, 2!, 3!, and 4!. So we have by induction that-

$$\int_{x=0}^{\infty} x^n exp(-x)dx = n! = \Gamma(n+1) = (n+1)\Gamma(n)$$

This result works for all numbers including fractions and complex numbers Thus n=-1/2 produces Γ(1/2)=sqrt(2).

A second integral whose value can be obtained by mathematical induction is-

$$\int_{x=0}^{\infty} x^{2n} exp(-x^2)dx = [(2n-1)!\, sqrt(\pi)]/[(n-1)!\, 2^{2n}]$$

We do this by noting that-

$$\int_{x=0}^{\infty} x^2 exp(-x^2)dx = \{[1]sqrt(\pi)]/[2\hat{}2]\}$$

$$\int_{x=0}^{\infty} x^4 exp(-x^2)dx = \{[1\cdot 3]sqrt(\pi)]/[2\hat{}3]\}$$

and-

$$\int_{x=0}^{\infty} x^6 exp(-x^2)dx = \{[1\cdot 3\cdot 5]sqrt(\pi)]/[2\hat{}4]\}$$

Also we can apply the identity-

[1·3·5·….(2n-1)]=[(2n-1)!]/[2^(n-1)(n-1)!]

This produces the above result. For n=2 we get an integral value of (3/8)π.

We have shown via several different examples that both series and certain integrals may be evaluated by mathematical induction. When applicable it represents the easiest way to determine the series and integral values. One can always check one's results by using an alternate solution approach. For the above definite integrals one can make variable substitutions to express their values as Laplace transforms.

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