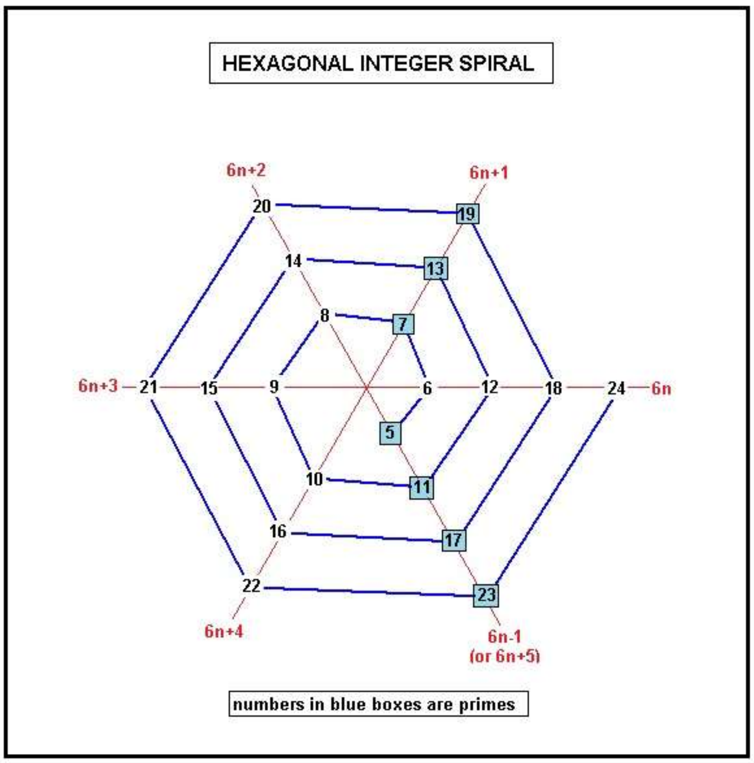
MORE ON HEXAGONAL INTEGER SPIRALS

About a decade ago we came up with a new way plot primes based upon the morphing of the Ulam Spiral. The new approach, finally being recognized by the mathematics community, plots all integers as points at the vertexes of a new integer spiral with primes lying strictly along two radial lines 6n±1 as shown-

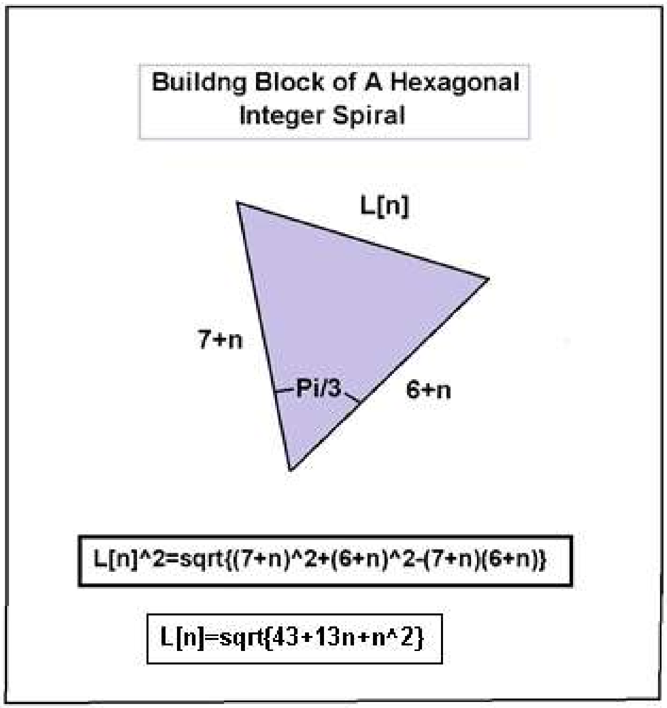


Such a spiral has its vertexes connected by straight lines not unlike what occurs for the Spiral of Cyrene. The primes in this integer spiral take on a very simple configuration with all primes(marked in blue), lying strictly along the radial lines 6n±1. Since there are also some composites such as 25 and 35 lying along these same two radial lines, one can make the more restrictive statement that-

A necessary but not sufficient condition that a number five or greater is a prime is that it satisfies 6n+1 or 6n-1 without exception.

We have tested this observation for primes as high as 100 digit length and find it to be true. Also this representation of integers has allowed us to find many results including new information on twin primes and aid in the factoring of large semi-primes. It is our purpose here to study integer spirals in more detail and especially find the closed form length of side-lengths L[n] of such spirals as a function of n.

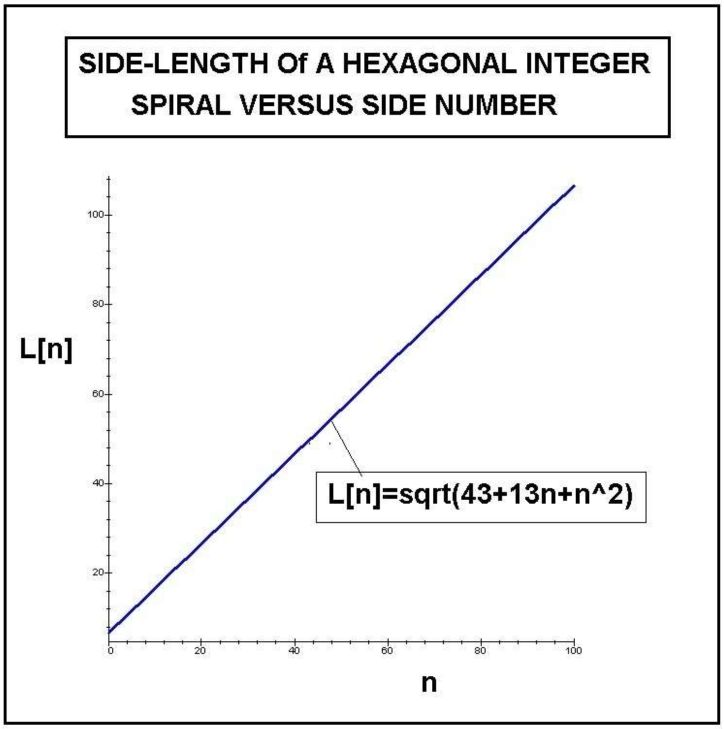
A closer inspection of the integer spiral shown above is that its basic building block is the non-right triangle shown-



Using the Law of Cosines, we find the length L[n] equals-

L[n]=sqrt{(7+n)^2+(6+n)^2-(7+n)(6+n)} = sqrt{n^2+13n+43}

A plot of L[n] versus n follows-



We see an almost linear relation between side-length L[n] and the value of n. Working out the first few side-lengths in detail, we have-

n	L[n]^2	n	L[n]^2
0	43	6	157
1	57	7	183
2	73	8	211
3	91	9	241
4	111	10	273
5	133	11	307

Here L[0] is the straight line distance from the x axis at 6 to the point(r,theta)=(7,Pi/3) in polar coordinates. This distance equals sqrt(43)=6.557438… Adding the square roots of L[n]^2 for n=0,1,2,3,4,5 yields the total distance around the first turn of the spiral, starting at x=6 and ending at x=12 .It equals 54.25888...We can estimate this number by a circle of radius r=9 equal to 2πr=56.548. Adding up the values of the roots of L[n]^2 from n=6 through 11 produces the length of 90.15185 for the second complete turn of the spiral going from x=12 to x=18.

Another interesting observation from the table is that-

57-43=14 , 73-57=16 , 91-73=18 , 111-91=20 , 133-111= 22

157-133=24 , 183-157=26 , 211-1]83=28 , 241-211=30 , 273-241=32

Note that the difference of the square of L[n-1] and L[n] goes up by units of two for each step. This means that in general-

L[n+1]^2=L[n]^2+2(6+n+1)

Thus, we have that L[12]^2=307+2(18)=343. So once L[0]^2=43 is given all other L[n]^2 will be known in closed form. What makes these values of L[n]^2 so easy to obtain is that the polar angle in the base triangle maintains the same value of sixty degrees (Pi/3 radians)for all triangles involved in the spiral construction.

One can use modular arithmetic to find out along which radial line a given integer lies along the hexagonal spiral. We have

N mod(6)=0 if lies along 6n

N mod(6)=1 if lies along 6n+1

N mod(6)=2 if lies along 6n+2

N mod(6)=3 if lies along 6n+3

N mod(6)-4 if lies along 6n+4

N mod(6)=5 if lies along 6n+5

So the number N=123456789 has a mod(6) value of 3 and so lies along the negative x axis. The number N=478193203 has mod(6) equal to 6n+1. So it lies along thr 6n+1 radial line and might possibly be a prime. To test whether this is so , one looks at the value of σ-N-1, where sigma is the sum function of number theory. If it is zero we have a prime. This is indeed the case for this 6n+1 prime since-

sigma(N)-N-1= 478193204- 478193203-1=0

Finally, we point out that semi-primes N=pq, such as encountered in cryptography, must also lie along the radial lines 6n±1. Take the example of the semi-prime-

N=pq=559846993

Its prime factors are p=14783 and q= 37851. Here we have N mod(6)=1 , p mod(6)=5, and q mod(6)=5.

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