FUNCTIONAL FORM OF NTH ORDER ALGEBRAIC EQUATIONS

Nth order algebraic equations may be written as-

$$y(x)=\prod_{n=1}^{N}(x-a_n)$$

, where a_n are the N roots of the equation. Thus a possible quadratic equation reads-

$$y(x)=(x-2)(x+3)=x^2+x-6$$

with the integer solutions x=2 and x=-3. A possible cubic equation is-

$$y(x)=x^3-6x^2+11x-6=(x-1)(x-2)(x-3)$$

with the three integer solutions x=1, 2, and 3. As is well known, solutions to all algebraic equations with N four or less can be expressed in radicals involving simple algebraic operations. However, as first shown by N.Abel in 1824, there exist no general solutions when N is five or greater. This does not mean, however, that algebraic equations greater than powers of N=4 don't exist. To get them for any integer N one works backwards by choosing values for a_n and then multiplying out the above product form.

Let us show this approach for N=1, 2, and 3. Here are the results-

N=1 $y(x)=x-a_1$

N=2 $y(x)=(x-a_1)(x-a_2)=x^2-(a_1+a_2)x+a_1a_2$

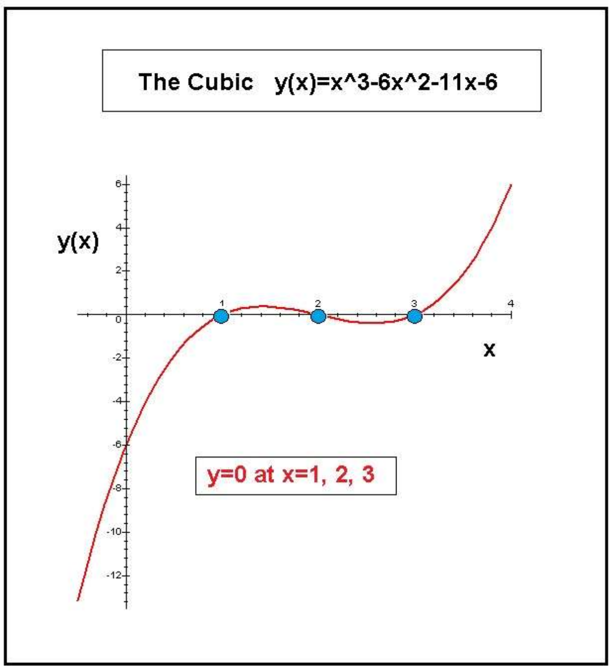
N=3 $y(x)=(x-a_1)(x-a_2)(x-a_3)=x^3-(a_1+a_2+a_3)x^2+(a_2a_3+a_1(a_2+a_3))x-a_1a_2a_3$

These expansions can be carried on to any larger integer N, with the equations y=y(x) becoming longer and longer. One of an infinite number of cubics is-

$$y(x)=x^3-6x^2+11x-6$$

It follows by setting $a_1=1$, $a_2=2$, and $a_3=3$ in N=3.

A plot of this curve looks as follows-



Here we have y=0 for x=1, 2, and 3. Note the odd symmetry about the vertical line x=2.

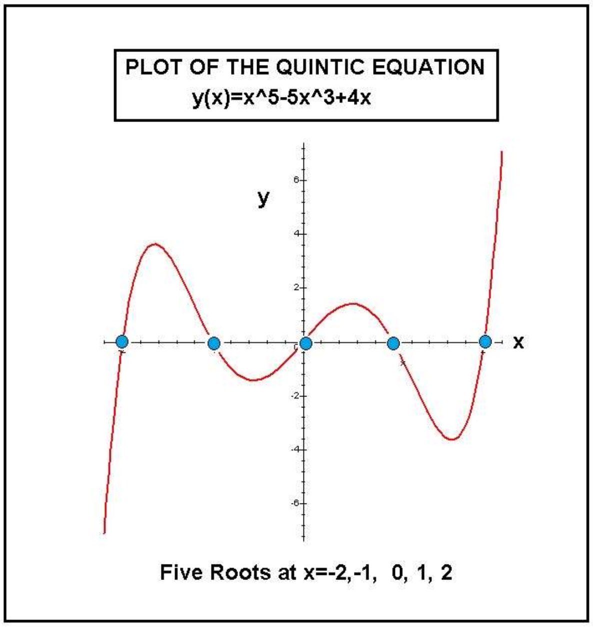
One can also easily construct a quintic equation . One of these is-

$$y(x)=\prod_{n=-2}^{2}(x-n) =(x+2)(x+1)x(x-1)(x-2)$$

That is-

$$y(x)=x^5-5x^3+4x$$

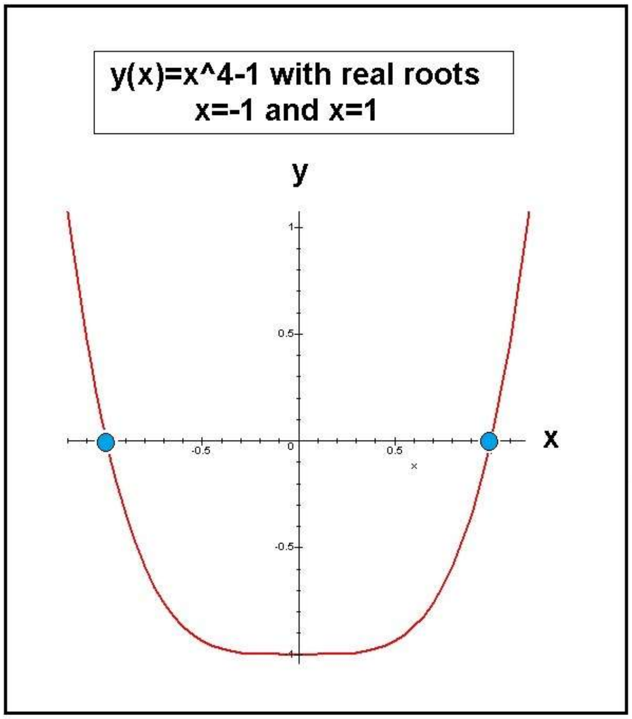
It yields y=0 at x=$\pm 2, x = \pm 1, and\ x = 0$. A plot of this last equation follows-



It is also possible to construct algebraic equations involving complex forms for the roots of y(x)=0. One such example is-

y(x)=(x-1)(x+1)(x-i)(x+i)=x^4-1

A graph for this y(x) follows-



Note here the even symmetry about the line x=0. Only the real roots y(x)=0 in this type of figure will show.

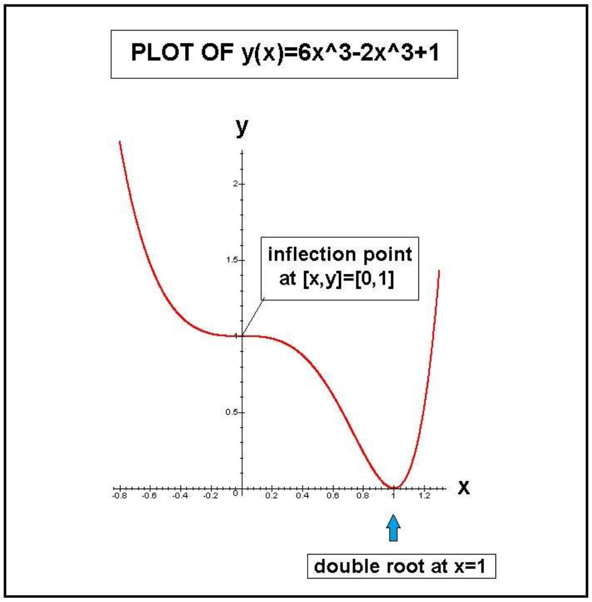
In all of the above algebraic equation examples we have the well known result that an Nth order algebraic equation has exactly N roots some of which may be complex and multiple. These days one can quickly find all roots of an algebraic equation by the simple MAPLE computer program-

solve(y(x)=0, x)

So if –

y(x)=x^6-2x^3+1

we get the six roots x=1, 1, $\left(\frac{1}{2}\right)[1 \pm i\ sqrt(3)]$, $(\frac{1}{2})[-1 \pm i$ sqrt(3)] . Only two of these are real. It is the double root at x=1. Here is the y(x) graph-



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