CONSTRUCTING PADE APPROXIMATES

A standard Maclaurin Series can be written as-
\[
f(x) = \sum_{k=0}^{N} c_k x^k = c_0 + c_1 x + c_2 x^2 + \ldots + c_N x^N \quad \text{with} \quad c_k = f(0)^{[k]} / k!
\]

As first pointed out by the French mathematician Henri Pade(1863-2053) this expansion can also be approximated by the polynomial quotient-
\[
R(x,m,n) = \frac{P(x,m)}{Q(x,n)} = \frac{\sum_{k=0}^{m} c_k x^k}{1 + \sum_{k=1}^{n} c_k x^k} = \frac{a_o + a_1 x + a_2 x^2 + \ldots a_m x^m}{1 + b_1 x + b_2 x^2 + b_3 x^3 + \ldots b_n x^n}
\]

Most advanced math computer programs call R(x,m,n) the Pade Approximant with a shorthand designation of pade(f(x),x,[m,n]). The evaluation of a_k and b_k coefficients are obtained by evaluating the identity-
\[
f(x) Q(x,n) = P(x,m)
\]

by setting the coefficients of x^k to zero. The larger m and n are taken the more accurate an approximation for f(x) will become. To find unique values for a_k and b_k will require m+n+1 algebraic equations and so f(x) must be expanded out to k=n+m starting with k=0.

We want in this note to develop several different Pade Approximates. Starting with one of the simplest forms, consider f(x)=exp(x) with m=n=1. This produces-
\[
(1 + x + \frac{x^2}{2})(1 + b_1 x) = (a_0 + a_1 x)
\]

and its three equations-
\[
a_0 = 1, \quad a_1 = b_1 + 1, \text{ and } (1/2) + b_1 = 0
\]

The Pade Approximate becomes-
\[
pade(exp(x),x,[1,1]) = \frac{1 + \frac{x}{2}}{1 - \frac{x}{2}}
\]
Note that m and n determine the highest powers of x present in P(x,m) and Q(x,n), respectively. Also the Maclaurin series contains x=m+n as its highest power. We next improve this Pade approximation by considering m=3 and n=2. This produces-

\[
(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120})(1 + b_1 x + b_2 x^2) = (a_o + a_1 x + a_2 x^2 + a_3 x^3)
\]

On solving for the coefficients we find-

\[
pade(\exp(x), x, [3,2]) = \frac{1 + \frac{3}{5} x + \frac{3}{20} x^2 + \frac{1}{60} x^3}{1 - \frac{2}{5} x + \frac{1}{20} x^2}
\]

A plot of exp(x), and pade(exp(x),x,[3,2]) follows-

We see that the Pade approximate is in good agreement with exp(1) up to about x=3. To get agreement above this value of x will require larger values for n and m.

Here is a Pade approximate for the tangent function showing its singularities at \((2n-1)\pi/2\)-

\[
pade(\tan(x), x, [5,5]) = \frac{x}{15} \left\{ \frac{945 - 105x^2 + x^4}{63 - 28x^2 + x^4} \right\}
\]

The graph follows-
A good estimate for the first infinity of \( \tan(x) \) is found by solving \( 63-12x^2+x^4=0 \). Its smallest root equals 1.570807884…. This is very close to \( \pi/2=1.570796327…. \) Note the approximation is very close to \( \tan(x) \) over the range -3.5<\( x <3.5 \). A most interesting result also found is that our earlier use of the TLK Method for approximating the values of all trigonometric functions (see the April 11, 2020 article-https://mae.ufl.edu//KTL-UPDATE.pdf ) yields a function T(5) identical with the above pade(\( \tan(x),x,[5,5] \)) result. This is a rather amazing result considering that a completely different approach involving Legendre Polynomials was used to find the given quotient approximation T(5).

We continue on to find an approximation for \( \pi \) using the function 4pade(arctan(\( \pi/4 \)),x,[50,50]). It produces the 38 digit accurate result-

\[
\pi=3.1415926535897932384626433832795028841…
\]

in a split second.

Consider next an approximation to \( \ln(2) \) using pade(\( \ln(1+x),x,[12,12] \)). It produces the 17 digit accurate result-

\[
\ln(2)=0.69314718055994530…
\]

when \( x=1 \).

Note that \( m \) and \( n \) need not always be equal in pade approximations. I usually prefer to look at cases where \( m \geq n \). Also the larger \( n \) and \( m \) become the closer one will get to an exact answer. Symmetry of the function \( f(x) \) also plays a role in any pade approximate. The quotient must have the same symmetry as the function \( f(x) \). One sees this clearly in the above case of pade(\( \tan(x),x,[m,n] \)).
As a last example let us look at the Gaussian \( f(x) = \exp(-x^2) \) for several different cases of \([m,n]\). It being an even function means that all powers of \( x \) in the pade approximate will be even. Here are three of such quotients-

\[
p\text{ade}(\exp(-x^2),x,[2,2]) = \frac{1 - \frac{1}{2}x^2}{1 + \frac{1}{2}x^2}
\]

\[
p\text{ade}(\exp(-x^2),x,[4,2]) = \frac{1 - \frac{1}{3}x^2 + \frac{1}{6}x^4}{1 + \frac{1}{3}x^2}
\]

\[
p\text{ade}(\exp(-x^2),x,[4,4]) = \frac{1 - \frac{1}{2}x^2 + \frac{1}{12}x^4}{1 + \frac{1}{2}x^2 + \frac{1}{12}x^4}
\]

A plot of two of these and the Gaussian follow-

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