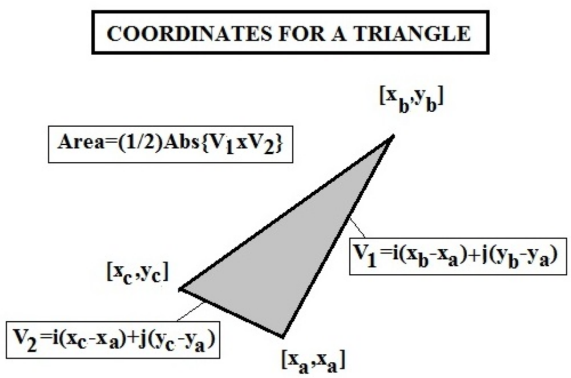
FINDING THE AREA OF ANY POLYGON

A polygon is any closed curve having a total of N straight line edges. Te length of the edges are equal for regular polygons but unequal when dealing with irregular polygons. One notices that the area a four sided N=4 polygon can be represented by superimposing the area f the two triangles formed by drawing a line from opposite vertexes. Likewise a pentagon with N=5 can always be broken up into T=3 triangles by using two lines going between non-neighboring vertexes. In general an N sided polygon has its area represented by T= N-2 triangles. Since one can always represent the area of any triangle by the coordinates of its three vertexes, it is therefore possible to express the area of any N sided polygon by simply adding up the areas of its sub-triangles. It is our purpose here to do so.

Our starting point for doing this is to look at the area for any of the sub-triangles of a given polygon of N sides. We let this triangle have the three vertex coordinates $[x_a,y_a]$, $[x_b,y_b]$, and $[x_c,y_c]$ as shown in the accompanying figure-



In the picture we also show two of the side length vectors emanating from ‘a’. Taking half of the absolute value of the cross-product of these two vectors yields the triangle area of-

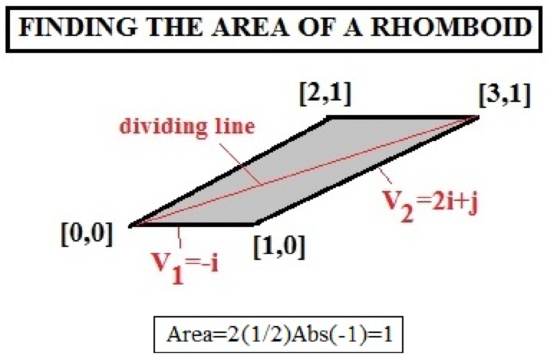
$$A = \frac{1}{2} Abs\{\begin{vmatrix} i & j & k \\ (x_b - x_a) & (y_b - y_a) & 0 \\ (x_c - x_a) & (y_c - y_a) & 0 \end{vmatrix}\}$$

$$= \frac{1}{2} Abs\{(x_b - x_a)(y_c - y_a) - (y_b - y_a)(x_c - x_a)\}$$

From this result we see that an equilateral triangle with vertexes at [-1,0], [1,0], and [0, sqrt(3)] has an area-

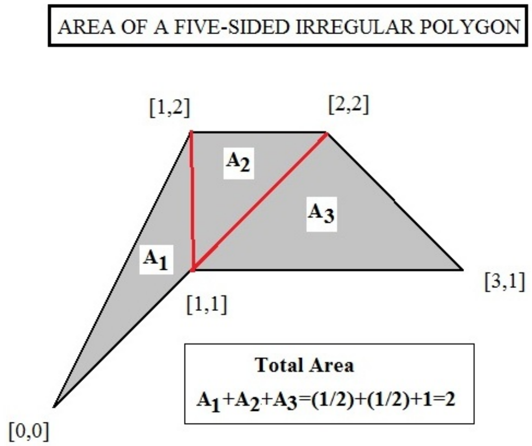
$$A = \frac{1}{2} Abs\{(1+1)(\sqrt{3}-0)-(0-0)(\sqrt{3}+1)\} = \sqrt{3}$$

With the above result for the area of a general triangle, we are now in a position to calculate the area of any polygon. Lets try a few cases. Start with the rhomboid having corners at [0,0], [1,0], [3,1] and [2,1]. We break the figure up into two triangles using a bisecting line going from[0,0] to [3,1]. One has the following picture-



Here we note that the two triangles are equal in area so the total area will be just twice that of one of the triangles shown. The lower triangle has an area of half of the cross-product of the V_1 and V_2 shown. This equals (1/2) so that the total Rhomboid area is 1. In this case, one could of course also get the answer by just subtracting the areas of two equal external triangles from a large rectangle of dimensions 1x3.

Consider next a five sided irregular polygon (N=5] with vertexes at [0,0], [1,1], [3,1], [2,2], and [1,2]. We can divide this closed polygon into three triangles T=3 by using just two dividing lines as shown here in red-



Two of these triangles have an obvious solution of A_2=1/2 and A_3=1. The thirdarea is generated by-

$$A_1 = \frac{1}{2} Abs \begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 1 & 2 & 0 \end{vmatrix} = \frac{1}{2}$$

This yields a total area for the gray shaded polygon of 2. Note that for this polygon one can combine A_1 and A_2 into a single triangle and thus reduce the required calculations.

Consider next a regular N sided polygon with each edge having a length of 1. In this case it is simplest if one determines the polygon's total area by adding up the N equal sizes of an isoscelis triangle of base length 1 and height h=1/(2tan(π/N). This produces a total polygon area of-

$$A_N = N\{\frac{1}{4\tan(\frac{\pi}{N})}\}$$

For a square this yields A_4=1, for a hexagon we have A_6= 3sqrt(3)/2, and for an octagon we get A_8=2/tan(π/8)=2/{sqrt(2)-1}. We also find that the ratio of the perimeter of the polygon compared to the circumference of the smallest circumscribed circle is exactly-

$$\text{Ratio} = \frac{N\sin(\frac{\pi}{N})}{\pi}$$

As N goes to infinity, this ratio approaches one. Archimedes first used this result to obtain accurate values for the constant π.

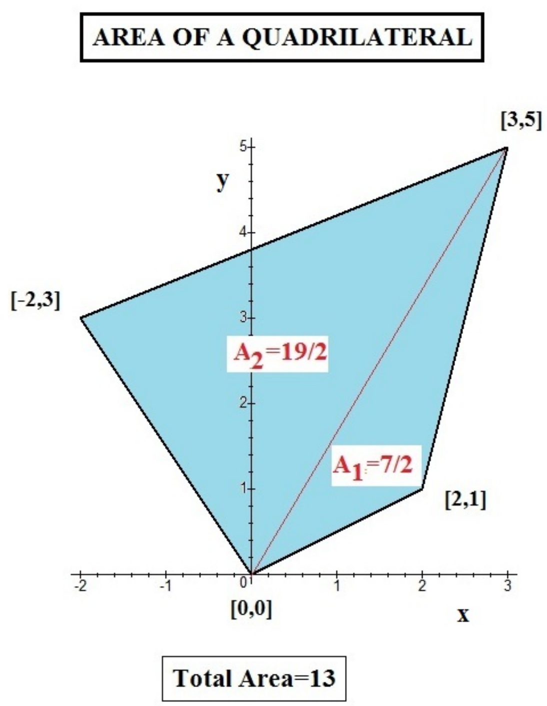
Finally let us calculate the area of an irregular four sided polygon N=4 with vertex coordinates of-

$$[x_a, y_a], [x_b, y_b], [x_c, y_c], [x_d, y_d]$$

Using the result for the area of a single triangle given at the beginning of this article and also replacing b by d, we get the total area of the four sided polygon to be-

$$Area = \frac{1}{2}Abs\{(x_b - x_a)(y_c - y_a) - (y_b - y_a)(x_c - x_a)\} + \frac{1}{2}Abs\{(x_d - x_a)(y_c - y_a) - (y_d - y_a)(x_c - x_a)\}$$

Here is the result for one particular quadrilateral-



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