

PROPERTIES OF PRIME NUMBERS

A prime number is defined as any integer greater than one which has no factors other than itself and one. Thus-

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 83, 89, 97,...

are prime numbers. Except for the integer $p_2=2$, all other even numbers are non-prime. The spacing $p_{n+1}-p_n$ between neighboring prime numbers goes as-

1, 2, 2, 4, 2, 4, 2, 4, 6, 2, 6, 4, 2, 4, 6, 6, 2, 6, 4, 2, 10, 6, 8,...

so that the difference, except for the first, are even numbers. This observation makes plausible the Goldbach conjecture (as yet unproven) that any even number can be represented as the sum of two primes. Thus $64=59+5=41+23=17+47$ etc. Also one notices that number of primes in a given interval decreases with increasing number n . As first noticed by both Gauss and Legendre the approximate number of primes N less than n goes as $n/\ln(n)$. This is referred to as the Prime Number Theorem and gives the estimate of $n/\ln(n)=145$ (to the nearest integer) at $n=1000$ compared to the actual larger number of 168. An integral approximation due to Riemann reads-

$$N(n) = \int_{x=0}^n \frac{1}{\ln(x)} dx$$

It gives a somewhat better estimate for the number of primes and does so as an upper limit. For $n=1000$ one finds $N(1000)=177.6$ by breaking the integration interval into 0 to 0.999 and 1.001 to 1000 and thus avoiding the infinity at $x=1$.

To determine whether a number is prime one can use the Sieve of Eratosthenes approach which consists of writing out all odd integers in sequence so that 3 becomes the first integer which is prime. Next cross out all terms divisible by 3. Follow this by dividing the next sequence by the next non-vanishing term 5 and again drop those terms divisible by 5. Continue this procedure indefinitely and you will be left with the prime numbers. Mathematically one has-

3,5,7,9,11,13,15,17,19,21,23,25,27,28...->3,5,7,11,13,17,19,23,25,28...

->3,5,7,11,13,17,19,23,28...->3,5,7,11,13,17,19,23,...

a procedure which is easy to automate by computer.

Although there is no known formula which will generate a general prime, the formula-

$N = (2^p - 1)$ with p equal to the primes 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107 etc

will generate prime numbers as first noticed by Mersenne. Note, however, not all Mersenne numbers are prime. For example, $2^{11}-1=2047=(23)(89)$ is not. Computer evaluation programs to check out Mersenne primes are much simpler than those using the Eratosthenes Sieve and hence account for the presently verified largest primes. The latest of these primes is the huge number $N = (2^{32,582,657}-1)$ which has some 9.8 million digits.

Next let us develop the relationship between the Riemann zeta function and the prime numbers first found by Euler. Start with the definition-

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{(e^t - 1)} dt = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \frac{1}{7^s} + \dots$$

and then notice that-

$$\left(1 - \frac{1}{2^s}\right) \zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} + \dots$$

and that-

$$\left(1 - \frac{1}{3^s}\right) \left(1 - \frac{1}{2^s}\right) \zeta(s) = 1 + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{11^s} + \dots$$

What is clearly happening is that one has a product function involving the primes on the left and an infinite series on the right which approaches unity as the procedure is applied an infinite number of times. This results in Euler's famous equality -

$$\zeta(s) = \prod_{n=1}^{\infty} \left(\frac{1}{1 - \frac{1}{(p_n)^s}} \right) = \left(\frac{1}{1 - \frac{1}{2^s}} \right) \left(\frac{1}{1 - \frac{1}{3^s}} \right) \left(\frac{1}{1 - \frac{1}{5^s}} \right) \dots$$

From it one can conclude that

$$\frac{\pi^2}{6} = \zeta(2) = \left(\frac{4}{3}\right)\left(\frac{9}{8}\right)\left(\frac{25}{24}\right)\left(\frac{49}{48}\right)\left(\frac{121}{120}\right)\left(\frac{169}{168}\right)\dots$$

and

$$\frac{\pi^4}{90} = \sum_{n=1}^{\infty} \frac{1}{n^4} = \left(\frac{2^4}{2^4-1}\right)\left(\frac{3^4}{3^4-1}\right)\left(\frac{5^4}{5^4-1}\right)\left(\frac{7^4}{7^4-1}\right)\dots$$

Finally lets look at an extension of the Mersenne prime formula(see Lucas and Catalan about 1870). Let $M0_p=2^p-1$ and look at-

$$M1_p = 2^{M0_p} - 1 \quad \text{with} \quad p = 2, 3, 5$$

This yields the primes 7, 127, 2147483647. Next take-

$$M2_p = 2^{M1_p} - 1 \quad \text{for} \quad p = 2 \text{ and } 3$$

which yields the prime numbers 127, and $(2^{127}-1)$ with possible other ones following for larger p. It is tempting to assume that if $M0_p$ is a Mersenne prime then so should the corresponding MN_p s be. This is however not always the case. For example the large number-

$$M1_{13} = 2^{M0_{13}} - 1 = 2^{2^{13}-1} - 1$$

factors on my MAPLE program. Thus one can conclude that the numbers MN_p are not always prime although some will be. It would be interesting to check the number-

$$M2_5 = 2^{M1_5} - 1 = 2^{2^{31}-1} - 1 = 2^{2147483647} - 1$$

as soon as computing power become available for factoring such billion digit numbers efficiently. The Lucas-Lehmer test for this number would appear to be impractical. If it should turn out to be prime, then it will be the largest Mersenne prime ever found by orders of magnitude. Note that the exponent of 2 in this number is about half of the Fermat number $2^{32}+1=4,294,967,297$ (one observes that $2(2^{31}-1)-(2^{32}+1)=-3$). Fermat thought 4,294,967,297 to be prime but Euler proved it to not be so.

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