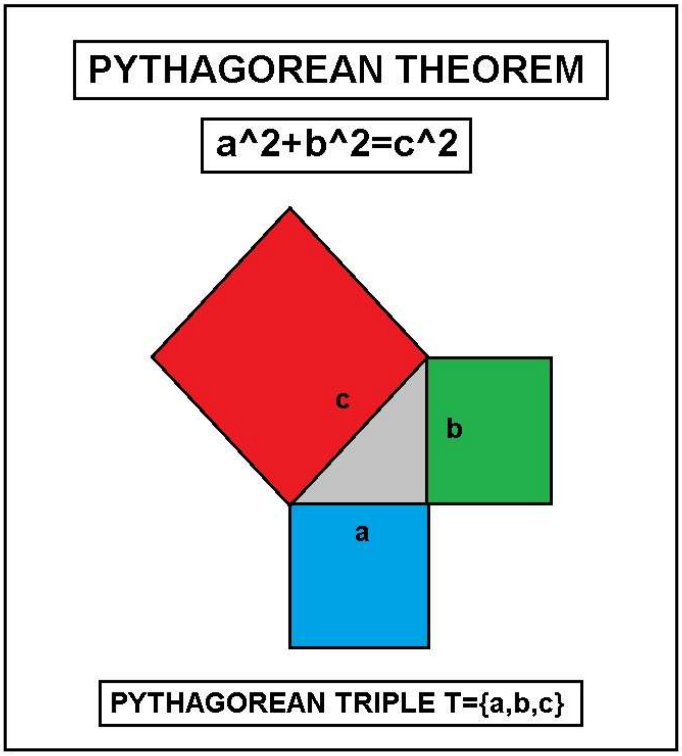
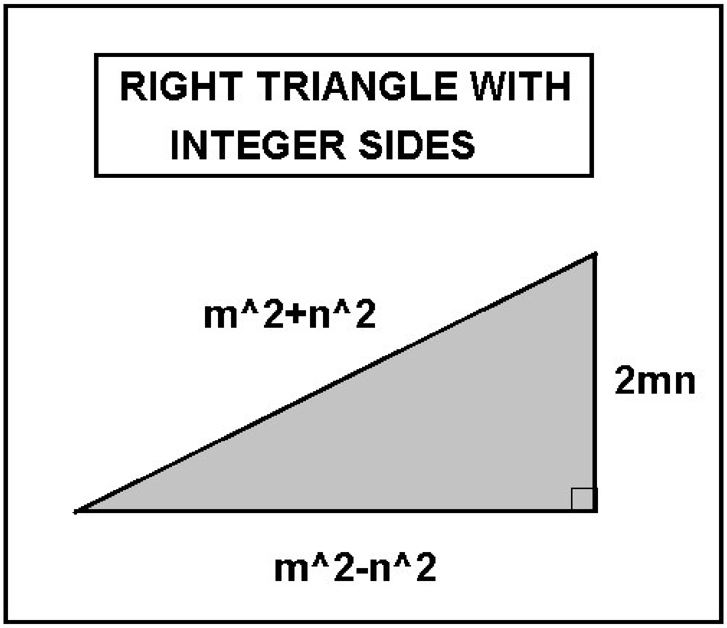
THE PYTHAGOREAN THEOREM AND ITS TRIPLES

One of the earliest mathematical theorems discovered is the Pythagorean Theorem (530BC) which states that the sum of the square of the shorter two sides of a right triangle equals the square of its hypotenuse. A geometric representation of this theorem follows-



There are many different proofs of this theorem. The earliest of which was given a thousand years prior to Pythagorous by the ancient Babyloneans in 1700 BC. We want in this article to examine some of its more important properties.

We start with a right triangle of sides a=m^2-n^2, b=2nm, and c=m^2+n^2 with integer m >n. Here is its picture-



Notice by rotating m^2-n^2 by 90 deg one recovers 2mn.This means we have a right triangle . Also the Pythagorem Theorem yields-

(m^2+n^2)^2=(m^2-n^2)^2+(2nm)^2

The three sides form the Pythagorean Triple-

T={m^2-n^2, 2mn,m^2+n^2}

The lowest triple involving all integer sides is T={3,4,5} corresponding to m=2 and n=1. We call this a base triple since multiplying by integer k we get an infinite number of additional larger triples T={3k,4k,5k}. Among these are T={159,212,265}.

Another base triple follows from m=3 and n=2. It reads T={5,12,13} . Muliplying each term in this triple by integer k produces an additional infinite number of triples given by T={5k,12k,13k}

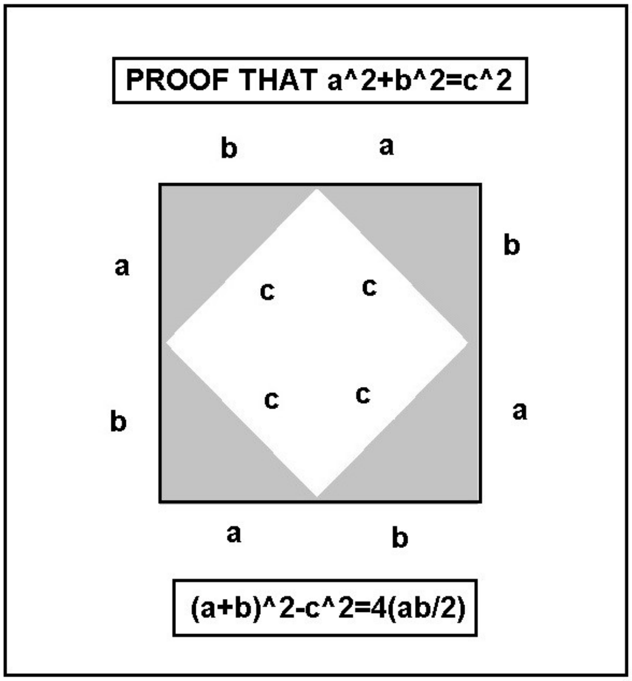
Next we look at m=4 and n=3. This yields T={7,24,25} as a base triple plus an additional number of extra triples T={7k,4k,25k}. Additional base triples will be established by using m=n+1 for m>4 . Using m=n-2, n-3 etc will recover triples of one of the lower base values. Thus m=4 and n=2 produces T={12,16,20} which is a k=4 multiplied version of T={3,4,5}.

Besides the right angle in the above triangle there are also the two other angles. These are given by-

A=arctan[(m^2-n^^2)/2mn] and B=arctan[2mn/(m^2-m^2)]

Thus we also have tan(A)tan(B)=1.The three angles associated with T={3,4,5}, where m=2 and n=1, equal A=36.86deg , B=53.13deg , and C=90deg.

The simplest way to prove the Pythagorean Theorem is to use the square within a rotated square approach. Here we get the following picture-



You see that the difference in area between the outer and inner square just equals four times the area of one of the triangles. This geometric proof is far simpler than that given by Euclid (325BC-265BC) in his book The Elements. If one relaxes the condition that a,b, and c all be integers, one can arrive at other equalities such as-

$$\sin(\theta)^2+\cos(\theta)^2=1 \quad , \quad c^2=a^2+b^2-2ab\cos(\theta)$$

and T={1,1,sqrt(2)} represents a right triangle with a=b=1 and a hypotenuse equal to c=sqrt(2).

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