It is well known that there are an infinite number of sequences $Y$ whose elements can be represented by a unique point function $y(n)$ which are also expressible as a difference equation. We wish in this article to examine such functions and show how they are derived.

Starting with the simple sequence-

$$Y = \{1, 3, 6, 10, 15, 21, \ldots\}$$

we can write down the following difference forms:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>3</th>
<th>6</th>
<th>10</th>
<th>15</th>
<th>21</th>
<th>zeroth difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td></td>
<td>first difference</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td>second difference</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>third difference</td>
</tr>
</tbody>
</table>

Since the differences stop with the third we know that the point function yielding the terms $y(n)$ of sequence $Y$ will have the form:

$$y(n) = An^2 + Bn + C$$

where the constants $A$, $B$, and $C$ are given by a solution of the matrix equation:

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$$

Solving, one has $A = B = 1/2$ and $C = 0$. Hence the sequence has individual element values of:

$$y(n) = \frac{n(n+1)}{2}$$

These $y(n)$ values represent the triangle numbers. So the seventh triangle number equals $y(7) = 28$. To get the difference equation for the elements, we can write $y(1) = 1$ and:

- $y(2) = y(1) + 2$
- $y(3) = y(2) + 3$
- $y(4) = y(3) + 4$
- $y(5) = y(4) + 5$

Generalizing, we find the difference equation:
\[ y(n) = y(n-1) + n \quad \text{subject to} \quad y(1) = 1 \]

One can also reverse the procedure by starting with a difference equation such as-
\[ y(n) = y(n-1) + 2n - 1 \quad \text{subject to} \quad y(1) = 1 \]

Writing out the elements we get-
\[ y(2) = 4, \quad y(3) = 9, \quad y(4) = 16, \quad y(5) = 25 \]

So the corresponding sequence reads-
\[ Y = \{1, 4, 9, 16, 25, \ldots\} \]

with the element values given by \( y(n) = n^2 \).

Consider next the sequence-
\[ Y = \{1, 2, 5, 12, 27, \ldots\} \]

Its elements have the corresponding difference equation form-
\[ y(n) = 2y(n-1) + n - 2 \quad \text{subject to} \quad y(1) = 1 \]

Writing out the differences we find-

\[
\begin{array}{cccccc}
1 & 2 & 5 & 12 & 27 & 58 & 121 \\
1 & 3 & 7 & 15 & 31 & 63 \\
2 & 4 & 8 & 16 & 32 \\
\end{array}
\]

One notices that here the second difference goes as \( 2^n \). It suggests the point function-
\[ y(n) = 2^n + An \]

At \( n = 3 \), this says \( 5 = 2^3 + A3 \), so that \( A = -1 \). Hence we have the point function-
\[ y(n) = 2^n - n \]

which generates all elements of the sequence \( Y \).

Take next the sequence-
\[ Y = \{1, 4, 11, 26, 57, \ldots\} \]
This sequence equals the elements encountered by looking at the elements along column two in a modified Pascal Triangle as discussed earlier at (http://www2.mae.ufl.edu/~uhk/MORE-PASCAL.pdf). Writing out the first few differences we find-

\[
\begin{array}{cccc}
1 & 4 & 11 & 26 & 57 \\
3 & 7 & 15 & 31 \\
4 & 8 & 16 \\
\end{array}
\]

This suggests a point function-

\[y(n)=2^{n+1}-(n+2)\]

So, for example, \(y(3)=11\), \(y(7)=247\) and \(y(10)=2036\). To get the corresponding difference equation, we write-

\[y(1)=1, \ y(2)=2y(1)+2, \ y(3)=2y(2)+3, \text{ and } y(4)=2y(3)+4\]

On generalizing we get the difference equation-

\[y(n)=2y(n-1)+n \quad \text{subject to } y(1)=1\]

Next, starting with the generalized difference equation-

\[y(n)=y(n-1)+f(n) \quad \text{subject to } y(1)=1\]

we find-

\[y(2)=1+f(2), \ y(3)=1+f(2)+f(3), \text{ and } y(4)=1+f(2)+f(3)+f(4).\]

This produces the solution-

\[y(n) = 1 + \sum_{k=2}^{n} f(k)\]

If we now take \(f(k)=k^2\), one finds the sequence and element values of-

\[Y = \{1, 5, 14, 30, 55, \ldots\}\]

and

\[y(n) = 1 + \sum_{k=2}^{n} k^2\]

If instead we take \(f(k)=(k-1)^2\), we get the sequence-

\[Y=\{1, 2, 6, 15, 31, 56, \ldots\}\]

Writing out the differences we have-
This suggests \( y(n) = An^3 + Bn^2 + Cn \). On solving the corresponding matrix equation for the constants \( A, B, \) and \( C \) we find the point function for the elements-

\[
y(n) = \frac{1}{8} \left[ 3n^3 - 7n^2 + 10n \right]
\]

Thus, for example, \( y(10) = 2400/8 = 300 \).

Starting with anyone of an infinite number of additional difference equations, one can also generate sequences with fractional elements and ones where \( y(\infty) \) reaches a finite limiting value. A good example for such a case starts with the difference equation-

\[
y(n + 1) = 1 + \frac{1}{y(n)} \quad \text{subject to} \quad y(1) = 1
\]

Its solution produces the fractional number sequence-

\[
Y = \{1, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \ldots \}
\]

Also it yields the interesting result –

\[
\lim_{n \to \infty} \left[ y(n) \right] = \sqrt{2} = 1.41421356...
\]

An additional well known constant \( y(\infty) \) is produced by the difference equation-

\[
y(n + 1) = 1 + \frac{1}{y(n)} \quad \text{subject to} \quad y(1) = 1
\]

Here we have the equivalent sequence-

\[
Y = \{1, 2, 3/2, 5/3, 8/5, 13/8, 21/13, 34/21, 55/34, 89/55\ldots\}
\]

The elements are easy to construct and seem to be heading towards 1.618 as \( n \) gets large. Indeed we find-

\[
y(\infty) = \phi = \frac{[1+\sqrt{5}]}{2} = 1.61803398\ldots
\]
with φ known as the Golden Ratio. The ancient Greeks were particularly fond of this irrational constant.

Another difference equation, which we first discovered back in 2012 while studying arctan expansions, is-

\[ y(n+1) = y(n) + \cos[y(n)]\{\cos[y(n)] - \sin[y(n)] \} \quad \text{subject to} \quad y(0) = 1 \]

In just eight iterations of this equation we find \( y(8) \) to yield a one-hundred digit accurate estimate for \( \frac{\pi}{4} \). We obtain-

\[ 4 \cdot y(8) \approx 3.1415926535897932384626433832795028841971693993751058209749445923 \]

in a split second.

As a final sequence consider the point function

\[ y(n) = \sum_{k=0}^{\infty} \frac{1}{k!} \]

Here we have the sequence, starting with \( n=0 \), of-

\[ Y = \{1, 2, 5/2, 8/3, 65/24, 163/60, \ldots\} \]

One recognizes at once that \( y(n) \) approaches exp(1)=2.7182818…

Also looking the individual terms in this sequence, we have the equivalent difference equation-

\[ y(n) = y(n-1) + \frac{1}{n!} \quad \text{subject to} \quad y(0) = 1 \]

Note that \( y(6) \) equals 163/60 + 1/720 = 1957/720.

U.H.Kurzweg
January 21, 2020
Gainesville, Florida