SUMMATION OF SERIES USING COMPLEX VARIABLES

Another way to sum infinite series involves the use of two special complex functions, namely-

$$F(z) = \pi f(z)\cot(\pi z) \quad , \quad G(z) = \pi f(z)csc(\pi z)$$

where f(z) is any function with a finite number of poles at z_1, z_2 , .. z_N within the complex plane and cot(π z) and csc(πz) have the interesting property that they have simple poles at all the intergers n= -∞,...,0,...+∞ along the real z axis.(click on the title to this section to see their graphs). One knows from Cauchy's residue theorem that the closed line contour enclosing all the poles of functions F(z) and G(z) equals 2π i times the sum of the residues. If we now demand that both F(z) and G(z) vanish on a rectangular contour enclosing all the poles , one has that-

$$\sum_{k=-\infty}^{+\infty} f(k) = -\sum_{n=1}^{N} Res[\pi f(z)\cot(\pi z), z = z_n]$$

and-

$$\sum_{k=-\infty}^{+\infty} (-1)^k f(k) = \sum_{n=1}^{N} Res[\pi f(z)csc(\pi z)], z = z_n]$$

where again the z_n refers to the location of the N poles of f(z). In deriving these results we have made use of the well known result that the residue for first order poles of g(z)/h(z) at the zeros of h(z) is simply $g(z_n)/h'(z_n)$.

Lets demonstrate this summation approach for several classical examples. Look first at the function $f(k)=1/(a^2+k^2)$ where f(z) has poles at z_1= ia and z_2= -ia . Plugging into the first residue formula above, we have-

$$\sum_{k=-\infty}^{+\infty} \frac{1}{k^2+a^2} = -\frac{\pi\cot(ia\pi)}{2ia} - \frac{\pi\cot(-ia\pi)}{-2ia} = \frac{\pi}{a}\coth(\pi a)$$

or, noting the even symmetry of the quotient cot(πa)/a, that-

$$\sum_{k=1}^{+\infty} \frac{1}{k^2+a^2} = -\frac{1}{2a^2} + \frac{\pi}{2a}\coth(\pi a)$$

If one takes the limit as **a** approaches zero(done by using the series expansions about a=0 for cosine and sine plus application of the geometric series) the famous result of Euler that the sum of the reciprocal of the square of all positive integers is equal to $\pi^2/6$ is obtained. As the next example look at f(k)=(1/(k^2m). Here we have just a single 2m th order pole at z=0 and one finds-

$$\sum_{k=1}^{\infty}\frac{1}{k^4}=\frac{\pi^4}{90} \quad , \quad \sum_{k=1}^{\infty}\frac{1}{k^6}=\frac{\pi^6}{945} \quad , \quad \sum_{k=1}^{\infty}\frac{1}{k^8}=\frac{\pi^8}{9450}$$

for m=2, 3 and 4, respectively.

Next we look at a series with alternating signs. For the following case we get-

$$\sum_{k=-\infty}^{\infty}\frac{(-1)^k}{(k^2+a^2)}=-Res[\frac{\pi csc(\pi z)}{(z^2+a^2)},\ z=\pm ia]=\frac{\pi}{a\sinh(\pi a)}$$

which allows one to state that-

$$\frac{1}{5}-\frac{1}{10}+\frac{1}{17}-\frac{1}{26}+..\frac{(-1)^k}{1+k^2}+...=\frac{\pi}{2\sinh(\pi)}$$

when a=1.

All of the above examples have involved even functions f(k). One now asks what about odd functions such as $f(k)=1/k^3$? Although the above residue formulas do not apply to odd functions, a modification is possible as we now show. Consider the function H(z)= π sec(πz)f(z), where f(z) is now an odd function, and then make a closed line contour integration of H(z) about the rectangular contour with corners at (N+1/2)(1+i), (N+1/2)(-1+i), (N+1/2)(-1-i), and (N+1/2)(1-i). This leads to-

$$(\frac{1}{2\pi i})\oint H(z)dz=Res[H(z),0]+\sum Res[H(z),n+1/2]$$

where the left integral vanishes as N goes to infinity, the residue for H(0) becomes $\pi^3/2$ when $f(z)=1/z^3$, and the residues H(n+1/2) become $1/[n+1/2)^3 \sin(\pi(n+1/2))]$.We thus have the interesting result that-

$$\frac{\pi^3}{32}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}=\frac{1}{1^3}-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+...$$