VARIATIONS ON THE SIGMA FUNCTION

It is known from our earlier work that f(N)=[sigma(N)-N-1]/N will vanish whenever N is a prime number. Here σ(N) is the sigma function of number theory and the parameter f(N) has been termed by us as the number fraction for any integer N. Since f(N) vanishes when N equals a prime p, it must also be true that-

$$\sigma(p)=1+p$$

If we replace p by p^2 and expand $\sigma(p^2)$, we arrive at-

$$\sigma(p^2) =1+p+p^2 \ .$$

Generalizing, we have a new identity for primes p given as-

$$\sigma(p^n)=\sum_{k=0}^{n} p^k$$

Thus we find-

$$\sigma(25)= 1+5+25=31 \quad \text{and} \quad \sigma(243)=1+3+3^2+3^3+3^4+3^5=364$$

Note that we can also make us of the properties of a finite geometric series to simplify these last two sigma values. The formula for doing this is to start with a finite geometric series, say-

S(p, n) = 1+p+p^2+p^3+…+p^n

Next we look at the infinite geometric series S(a,∞)=S(a,n)+a^(n+1)+a^(n+2)+….)=1/(1-a)

Manipulating things we find –

S(a,n)= 1/(1-a)-a^(n+1)/(1-a) = [1-a^(n+1)]/[1-a]

So if, as in the above example, we take a=5 and n=2, we get 31. Also if we take a=3 and n=5, we get 364. In addition, for all primes, we have -

$$\sigma(p^n)= S(p,n) = \left[\frac{1-p^{(n+1)}}{1-p}\right]$$

, where p is any prime number and n its integer power. With this modified formula, we see, for example, that-

σ(37^5)=σ(69343957)=71270178

Note that sigma is slightly larger than N. This is predicted by the above S(p,n) formula.

Expanding our discussion to σ(p^n q^m), we find the new and altered form for the sigma function is given as-

σ[(p^n)(q^m)]={[1-p^(n+1)][1-q^(m+1)]/[(1-p)(1-q)]

This will work for any primes p and q taken to integer powers n and m, respectively.

Let us try things out for p=3 and q=5 with powers n=2 and m=3. We have-

σ(1125)=(1-27)(1-625)/(8)= 2028

There is also no major difficulty to go to much larger values of N=(p^n)(q^m).Take the case

N=127^12*719^18=

464263646685280795614768732551024239000214091881135808276556208O5

747601393761. This yields in a split second that-

σ(N)=

468600017351693917075334673657770860994607417331048616864023427823

94796330161

Note that for such large Ns, which are often encountered in cryptography, the values of N and σ

lie fairly close to each other.

We can also use the above results to factor semi-primes N=pq. One finds that-

f(pq)=(p+q)/(pq) and σ(pq)=(1+p)(1+q)

Upon letting the average value of p and q be given by S=(p+q)/2, we find the semi-prime N=pq factors into its two prime components as-

$$[p,q]=S\mp\sqrt{S^2-N}$$

Take the case of N=1769 where f(N)=[sigma(N)-N-1]/N=90/1769=(p+q)/1769. This produces S=90/2=45 so that –

$$[p,q]=45\mp\sqrt{2025-1769}=[29, 61]\ .$$

Such solutions can be obtained quickly as long as σ(N) can be evaluated on one's computer. With my MAPLE program, I have no problem finding the values of σ(N)as long as N stays below about twenty digits or so. Future attempts to find values of σ(N) beyond this , should make it possible to factor semi-primes for Ns falling into the one hundred digit range such as encountered in modern day electronic cryptography.

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