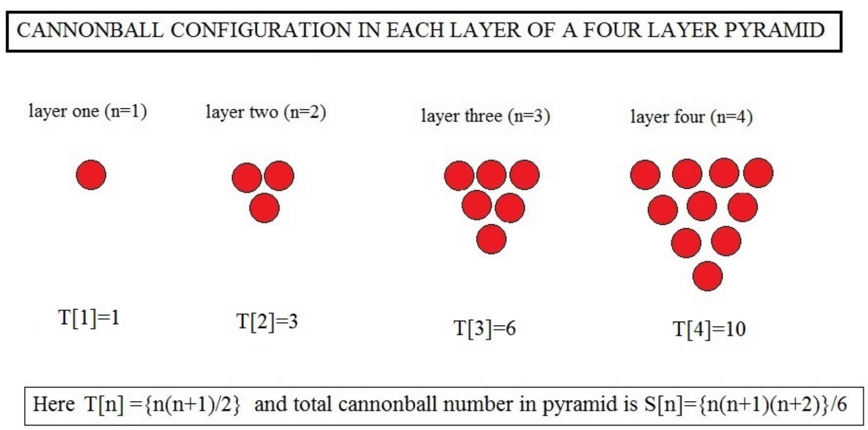
STACKING OF CANNONBALLS

It is well known that cannonballs can be stacked into stable pyramids where the base can be either a triangle or a square. The famous astronomer J.Kepler first looked at this problem in the sixteen hundreds and concluded that optimum stacking whichleaves a minimum of free space between the cannonballs is one where subsequent layers are so arranged that the balls in the upper layer neatly fit into the scallop spaces left by the balls in the lower layer. We want here to look at the details of such stackings and show how it naturally leads to formulas giving not only the number of cannonballs in a layer but also the total number of balls in the resultant pyramid.

Triangular Base:

We begin by looking at the first few layers of cannonball stacking with a triangular base. Here we have the following picture-



We see that the number of balls in each layer goes as-

T[n]={1, 3, 6,10, …}

These numbers, referred to as the **triangular numbers**, are generated by the addition formula T[n]=n(n+1)/2. For the four layer pyramid considered here, we find the sum of the cannonballs to equal S[4]=1+3+6+10=20. Note that we can write-

$$T[n+1]=\frac{(n+2)}{n}T[n]$$

This formula may be used to generate any of the triangle numbers T[n]. Thus we have T[5]=15,T[6]=21, and T[7]=28. To calculate the number of cannonballs in a stack with n layers, we write-

S[n]=1+3+6+10+15+21+28+……n(n+1)/2

This allows us to say that-

S[1]=1
S[2]=T[1]+T[2]
S[3]=T[1]+T[2{+T[3]

Thus we have-

$$S[n] = (1/2)\sum_{k=1}^{n} k(k+1) = \frac{n}{6}(n+1)(n+2)$$

From this formula we have-

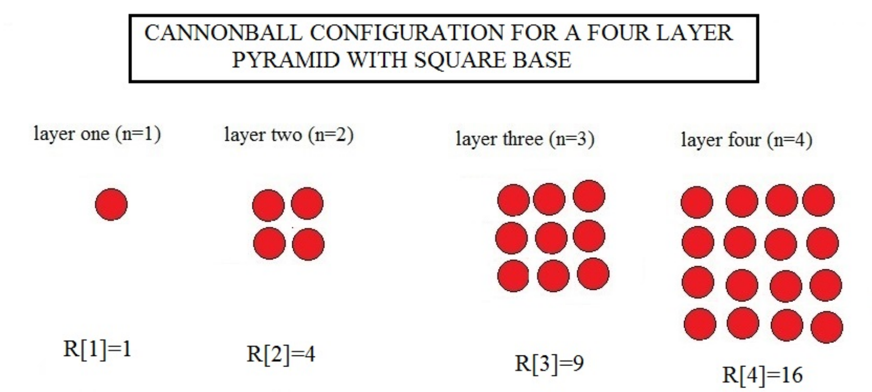
S[n]={1, 4, 10, 20, 35, 56, 84, 120, …}

Thus an eight layer cannonball pyramid with triangular base will contain precisely 120 cannonballs. A generating formula for S[n] is-

$$S[n+1] = \frac{(n+3)}{n} S[n] \quad subject \quad to \quad S[1] = 1$$

Square Base:

The second type of cannonball stacking involves a square base from which a four sided pyramid can be constructed. In this case a four layered pyramid will have the following layer configurations-



Here $R[n]=n^2$ and the total cannonball number in the pyramid is Q[n]={n(n+1)(2n+1)}/6

This time the number of balls in each layer is simply n^2. The sum of the cannonballs in the entire n layered pyramid becomes-

Q[1]=1
Q[2]=5
Q[3]=14
Q[4]=39

That is-

$$Q[n+1\}=Q[n]+(n+1)^2=\sum_{k=1}^{n+1}(k)^2=(\frac{1}{6})(n+1)(n+2)(2n+3)$$

This may also be written as the cubic-

$$Q[n]=(\frac{n}{6})\{n+1)(2n+1)$$

In searching the internet for some stacked cannonball images we came across the following-

STACKED CANNONBALLS WITH SQUARE BASE AT MONACO PALACE

four layers n=4 with Q[4]=30

z

We have there four layers to yield a total number of Q[4]=1+4+9+16=30 cannonballs. Note the holes in some of the balls showing. These balls were filled with explosive powder attached to a fuse during actual combat. The life of a cannoneer must have been pretty short due to premature explosions in the barrel.

Angles associated with Stacks:

There are three basic angles associated with the stable stack of cannonballs. These are the corner angle ψ at the base, the slant angle θ between the pyramid vertex and one of the base corners, and finally the minimum angle φ from the vertex to the middle of one of the base lines. As is obvious the corner angle for stable stacks must be ψ=60 deg for a triangular base and ψ=90 deg for a square base. The simplest way to find the slant angle θ is to consider a two stack configuration were all four or five balls touch each other and maintain the distance of twice the radius from each other. For an equilateral triangle base we can place the centers of the four balls at-

[x,y,z]={[0,0, sqrt(2)],[0,1,0],[sqrt(3)/2,-1/2,0],[-sqrt(3)/2,-1/2,0]}

to make a two layer stack. The distance between ball centers is here sqrt(3) between each of the balls. For them to just touch the radius of each must be sqrt(3)/2. Now the angle the center of the ball in layer one makes with any of the second layer centers will be-

θ=arcos[sqrt(2/3)]=35.26 deg

That is, the slant angle along any of the three edges will be 35.26 deg so that their steepness is 90-35.26=54.74. The minimum slant angle is given by-

φ=arcsin[1/(2sqrt(2))]=20.70 deg

Going next to the square base, we can place the four balls in layer two at [1,0,0],[0,1,0],[-1,0,0] and [0,-1,0]. The single ball in layer one will be located at [0,0,1]. The distance between the ball centers will be sqrt(2) for all cases. This means the balls have radius r=sqrt(2)/2 each. The slant angle is now easy to calculate and reads-

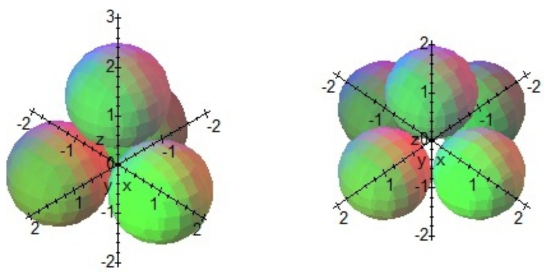
θ=arctan(1)=45deg

For the smallest angle we have-

φ=arctan[1/sqrt(2)]=35.26 deg

We can thus conclude that the slope for the triangular base of the slanted edges will be larger than that for a square base. A summary of these results is found in the following graphics rendering-

COMPUTER GRAPHICS OF A TWO LAYER CANNONBALL STACK



Triangular Base	Square Base
θ =arccos[sqrt(2/3)] = 35.26 deg	θ =arcsin[1/sqrt(2)] =45 deg
φ =arcsin{1/[2sqrt(2)]} =20.70 deg	φ =arctan[1/sqrt(2)] =35.26 deg

What struck us as interesting is that the Egyptian pyramids, especially the Great Pyramid of Cheops(or Khufu) at Giza, have similarities with stacked cannonballs. They are both extremely stable structures. The Great Pyramid itself has survived some 4000 years without structural damage despite of earthquakes and Vandals. It remains as the only extant of the seven wonders of the ancient world. It has a square base with ψ=90deg. Side length of b=756 ft and original height estimated as H=481 ft . Its slant angle equals-

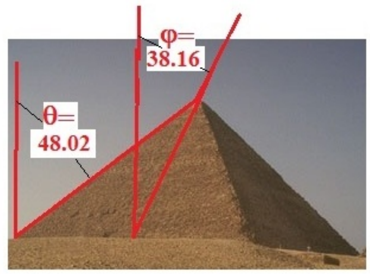
$$\theta=\arctan[b/(H\mathrm{sqrt}(2)]=48.08 \text{ deg}$$

and the smallest slant equals-

$$\varphi\text{-}\arctan[b/(2H)]=38.16 \text{ deg}$$

Here is a picture of these angles for the Great Pyramid-

ANGLES ASSOCIATED WITH THE GREAT PYRAMID AT GIZA



base length b=756 ft **pyramid height H=481 ft**

slant angle θ =arctan{(b/[Hsqrt(2)]} **smallest angle φ= arctan{b/[Hsqrt(2)] }**

The resultant slant and smallest angles are very close to that of a square base cannonball stack. One could speculate that the ancient Egyptian pyramid builders were familiar with what makes for a stable pyramid from models resembling stacked cannonballs. There is evidence that they learned about structural stability after several earlier building attempts using a too steep pyramid angle (θ<<45 deg) led to pyramid collapse.

U.H.Kurzweg
May 5, 2017