DERIVATION OF THE IMPROVED STIRLING FORMULA FOR N!

We have shown in class, by use of the Laplace method, that for large n, the factorial equals approximately

$$n! \cong \sqrt{2\pi n}\, n^n \exp(-n)$$

This is referred to as the standard Stirling's approximation and is quite accurate for n=10 or greater. However, as n gets smaller, this approximation requires correction terms which can be obtained by taking more terms in the Taylor expansion for h(t) appearing in the exponent of the integrand. Since you may be interested in how this is achieved (especially in view of the latest homework assignment), let me give you a quick derivation.

Our starting point is the standard integral for n! , namely,

$$n! = \int_o^{\infty} t^n \exp(-t)dt = \int_0^{\infty} \exp n[\ln(t) - (t/n)]dt$$

Here h(t)=[ln(t)-t/n)] which clearly has a maximum at t_1=n. Expanding h(t) in a Taylor series about this maximum point for N terms, we get the approximate integral

$$n! \cong n^n \exp(-n) \int_{-\infty}^{\infty} (dz/du) \exp - (nu^2)du$$

with $u^2=z^2/2-z^3/3+z^4/4-z^5/5+\ldots$and z=(t-n)/n. On inverting this series we find

$$z = \sqrt{2}u + (2/3)u^2 + (\sqrt{2}/18)u^3 + O(u^4)$$

Taking the derivative dz/du, then yields

$$n! \cong \sqrt{2n}\, n^n \exp(-n) \int_{-\infty}^{+\infty} [1 + (4/3\sqrt{2})u + (1/6)u^2 + ..] \exp(-nu^2)du$$

This integral can be evaluated term by term via the gamma function and by symmetry one sees that all of the integrals involving odd powers of u must vanish. The result through u^2 thus yields the desired improved form

$$n! = \sqrt{2\pi n}\, n^n \exp(-n)[1 + 1/(12n) + ..]$$

One can carry this procedure further (if you have the patience), with the next two terms in the asymptotic series being $1/(288n^2)$ and $-139/(51{,}840n^3)$.