

DERIVATION OF THE STOKES DRAG FORMULA

In a remarkable 1851 scientific paper, G. Stokes first derived the basic formula for the drag of a sphere(of radius $r=a$ moving with speed U_0 through a viscous fluid of density ρ and viscosity coefficient μ) . The formula reads-

$$F = 6\pi\mu a U_0$$

It applies strictly only to the creeping flow regime where the Reynolds number $Re = \rho a U_0 / \mu$ is less than unity and is thus mainly concerned with spheres of very small diameter when the surrounding fluid is either a gas or a liquid. The formula has found a wide range of applications ranging from determining the basic charge of an electron to predicting the settling velocity of suspended sediments. In the bio-fluids area it is encountered when studying the settling rates of blood cells when centrifuged and in the determination of sedimentation rates of contaminants entering the lungs.

Most introductory texts (and even more advanced fluid books) generally do not give a full derivation of Stokes's Drag Law because of the mathematical complexity involved. We help remedy this situation here by giving a full derivation .Let me know if there is anything that is still not clear after you read this development.

First one introduces a spherical coordinate system with the sphere of radius $r=a$ placed at its center. The viscous and incompressible flow about the sphere will have only radial and polar angle dependent velocity components. The fact that the divergence of this velocity field is zero allows one to introduce the stream function $\psi(r ,\theta)$ defined by-

$$V_r = \frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta} \quad , \quad V_\theta = -\frac{1}{r \sin \theta} \frac{\partial \psi}{\partial r}$$

Next one looks at the two momentum equations for a steady creeping flow. Using the fact that $\nabla^2 \mathbf{V}$ can be replaced by $-\text{curl}(\text{curl } \mathbf{V})$ for an incompressible fluid, these equations can, after a little manipulation , be re-written as-

$$\frac{\partial p}{\partial r} = \frac{\mu}{r^2 \sin \theta} \frac{\partial(Q\psi)}{\partial \theta}, \quad \frac{\partial p}{\partial \theta} = -\frac{\mu}{\sin \theta} \frac{\partial(Q\psi)}{\partial r}$$

where Q a differential operator defined as-

$$Q = \frac{\partial^2}{\partial r^2} + \frac{\sin \theta}{r^2} \frac{\partial}{\partial \theta} \left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \right]$$

Eliminating p between these two momentum equations, one finds the fourth order PDE

$$Q^2 \psi = 0$$

Making the substitution

$$\psi = f(r) \sin^2 \theta$$

allows one to reduce this equation to a 4th order ODE of the standard Euler type

$$r^4 f'''' - 4r^2 f''' + 8rf'' - 8f' = 0$$

Next, a simple substitution $f=r^k$ leads to a 4th order polynomial in k whose roots are k=-1, 1, 2, and 4. Thus the general solution for f becomes

$$f(r) = \frac{A}{r} + Br + Cr^2 + Dr^4$$

We can evaluate the four constants A, B, C, and D from the fact that both velocity components vanish at the sphere surface at $r=a$ and that V_r goes as $U_0 \cos \theta$ as r gets large. This leads to the desired exact stream function for Stokes flow of

$$\psi(r, \theta) = U_0 \left[\frac{a^3}{4r} - \frac{3ar}{4} + \frac{r^2}{2} \right] \sin^2 \theta$$

From this last result, and use of the definitions for the velocity components and radial pressure gradient given above, one obtains the explicit values-

$$V_r = U_0 \left[\frac{a^3}{2r^3} - \frac{3a}{2r} + 1 \right] \cos \theta \quad , \quad V_\theta = U_0 \left[\frac{a^3}{4r^3} + \frac{3a}{4r} - 1 \right] \sin \theta \quad ,$$

$$p = - \left[\frac{3a\mu U_0}{2r^2} \right] \cos \theta$$

The drag force can now be calculated by integrating the shear stress and normal stress over the entire surface of the sphere. In this calculation p is the normal stress and the shear stress has the form-

$$\tau_{r\theta} = -\mu \left[r \frac{\partial}{\partial r} \left(\frac{V_\theta}{r} \right) + \frac{1}{r} \left(\frac{\partial V_r}{\partial \theta} \right) \right]$$

Carrying out the integration involving the pressure first, one has that the force in the z direction due to the pressure field is

$$F_{pressure} = 2\pi a^2 \int_0^\pi p \sin \theta \cos \theta d\theta = 2\pi a \mu U_0$$

Next the force in the z direction due to viscous shear stress is

$$F_{shear} = 2\pi a^2 \int_0^\pi \tau_{r\theta} \sin^2 \theta d\theta = 4\pi a \mu U_0$$

Adding things together we arrive at the famous Stokes Drag Law first derived by him over 150 years ago! Note that in this problem two thirds of the force on the sphere is due to viscous shear and only one third is due to pressure drag.

When dealing with small spheres dropping with constant speed in a gravity field one must equate the Stokes drag to the effective downward force equal to the sphere weight minus the buoyancy force. This yields

$$U_0 = \frac{2ga^2}{9\mu} [\rho_{sphere} - \rho_{fluid}]$$

From this result one can infer that a 20micron diameter iron sphere dropping in quiescent water will have a terminal downward velocity of just 1.5 mm/sec. The corresponding Reynolds number will be $Re=0.015$ and so lies well within the creeping flow regime. Since the terminal velocity is proportional to the square of the sphere diameter one could segregate out different size spheres from a mixed batch injected horizontally at constant speed.

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