SUMMATION OF SEQUENCES OF NUMBERS

Often one is interested in finding the sum of a sequences of numbers. Sometimes these follow a distinct functional form which can be represented by a single formula. One such sequence is –

$$S[n] = \{1, 3, 6, 10, 15, 21, 28, \ldots\}$$

To generate a functional form we first write down an array of numbers representing various differences. This array reads-

<table>
<thead>
<tr>
<th>1</th>
<th>3</th>
<th>6</th>
<th>10</th>
<th>15</th>
<th>21</th>
<th>28</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
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<td>1</td>
<td>1</td>
<td></td>
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<tr>
<td>0</td>
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<td>0</td>
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</tr>
</tbody>
</table>

This suggests we have a form $$S[n] = A + Bn + Cn^2$$ with no cubic term possible. It means we must have –

1 = A + B + C
3 = A + 2B + 4C
6 = A + 3B + 9C

On solving, this produces the formula-

$$S((n]) = n(1+n)/2$$

From this result we see that $$S[n]$$ is just the sum of the first n integers. That is, $$S(5) = 15$$ and $$S(7) = 28$$. The first hundred integers sum to $$S(100) = 5050$$. It is our purpose in this article to find general formulas for difference equations for a variety of other infinite sequences.

Let us begin with the sequence-

$$S[n] = \{1, 5, 14, 30, 55, 91, 140, \ldots\}$$

Here the corresponding difference array reads-

<table>
<thead>
<tr>
<th>1</th>
<th>5</th>
<th>14</th>
<th>30</th>
<th>55</th>
<th>91</th>
<th>140</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>9</td>
<td>16</td>
<td>25</td>
<td>36</td>
<td>49</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>7</td>
<td>9</td>
<td>11</td>
<td>13</td>
<td></td>
<td></td>
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<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
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</tr>
</tbody>
</table>

This form suggests we have a $$S(n)$$ formula which is a cubic-
\[ S(n) = A + Bn + Cn^2 + Dn^3 \]

The corresponding equations read-

\[ 1 = A + B + C + D, \quad 5 = A + 2B + 4C + 8D, \quad 14 = A + 3B + 9C + 27D, \quad \text{and} \quad 30 = A + 4B + 16C + 64D \]

On solving we have \( A = 0, \ B = \frac{1}{6}, \ C = \frac{1}{2}, \ \text{and} \ D = \frac{1}{3}. \) It produces the formula-

\[ S(n) = \frac{(n + 3n^2 + 2n^3)}{6} \]

One sees that \( S(2) = 5, \ S(10) = 2310, \ \text{and} \ S(100) = 230100. \) That is, this series represents the sum of the squares of the first \( n \) integers.

From the two sequences given above, we can conclude that a formula representing the sum of the first \( n \) integers taken to the \( pth \) power each produces a polynomial formula in powers through \( p+1. \)

Often one encounters formulas for \( S[n] \) which contain more than just a polynomial in \( n. \)

Let us demonstrate the sequence governed by the formula-

\[ S[n] = (1+n)2^{n-2} \]

Writing out the sequence, we find-

\[ S[n] = \{1, 3, 8, 20, 48, 112, \ldots \} \]

Here a difference array of \( S[n] \) shows no regularity meaning we are dealing with a function other than just a polynomial. So we look at any other regularities one can find. We note that –

\[ S[n+1] = 2S[n] + 2^{(n-1)} \] with \( S[1] = 1 \)

This represents a difference equation whose solution yields \( S[n]. \) Using the one line computer program-

\[ S[1] := 1; \ \text{for} \ n \ \text{from} \ 1 \ \text{to} \ 10 \ \text{do} \ S[n+1] := \text{evalf}(2*S[n] + 2^{(n-1)}) \od; \]

We find-


These each satisfy the formula \( S[n] = (1+n)2^{(n-2)}. \) Here \( S[100] = 101*2^{98}. \)

Several other sequences are recoverable from the modified Pascal Triangle which we discovered about five years ago. Here is its form-
Note that the sum of the elements in each row $m$ equal exactly $m!$.

Let us first consider the sequence given by-

$$S[n] = \{1, 4, 11, 26, 57, 120, \ldots\}$$


$$S[n+1] = 2S[n] + n + 1 \quad \text{subject to} \quad S[1] = 1$$

Running our computer search program yields the first ten values of $S[n]$ given by-

$$S[n] = \{1, 4, 11, 26, 57, 120, 247, 502, 1013, 2036, \ldots\}$$

It is also possible to run things backwards starting with say –

$$S[n] = n2^{n-1}$$

This produces the sequence-

$$S[n] = \{1, 4, 12, 32, 80, 192, 448, 1024, 5120, \ldots\}$$

From it we also have the difference equation-

$$S[n+1] = 2S[n] + 2^n \quad \text{subject to} \quad S[1] = 1$$

The closed form solution to this equation is $S[n] = n2^{n-1}$.

As a final $S[n]$ formula consider-

$$S[n+1] = -1 + n^2 + 2^n \quad \text{subject to} \quad S[1] = 1$$

Writing out the first ten elements we get-

$$S[n] = \{1, 5, 12, 23, 40, 67, 112, 191, 336, 611, \ldots\}$$
From this we also have the difference equation-

\[ S[n+1] - S[n] = 1 + 2n + 2^{(n-1)} \quad \text{subject to} \quad S[1] = 1 \]

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