SUMMATION OF FINITE SERIES

It is well known that certain finite series can be represented as simple functions of n. One of these, representing the sum of the first n integers, reads-

$$S(n)=\sum_{k=1}^{n} k = 1+2+3+\ldots+(n-1)+n$$

Rearranging yields-

$$S(n)=\{(1+n)+(2+n-1)+(3+n-2)+\ldots\}=n(n+1)/2$$

So the sum of the first 100 integers equals S(100) = 5050. Note that S(n) is here a quadratic function of n. One would therefore expect the sum of the first n squares to be a cubic in n. Indeed one finds-

$$S(n)=\sum_{k=1}^{n} k\hat{}2 = n/6+n\hat{}2/2+n\hat{}3/3= [n(1+n)(1+2n)]/6$$

Thus the sum of the squares of the first ten integers will be-

$$S(10)=10/6+100/2+1000/3=385$$

In view of the above results it is also clear that the sum of a finite series involving the pth integer power of the integers will be represented by a p+1 power polynomial. This fact was first recognized by the Bernoulli brothers over 300 years ago.

In addition to having a finite series represented by simple polynomials, there are many others where a polynomial representation won't work. One of these series is the finite geometric series-

$$G(p,n))=\sum_{k=0}^{n} p\hat{}k = 1+p+p\hat{}2+p\hat{}3+\ldots..+p\hat{}n$$

Here we find that –

$$G(p,n)=[p\hat{}(n+1)-1]/[p-1]$$

If p=3 and n=4, we get –

$$G(3,4)= 1+3+9+27+81=121$$

Notice here that p need not be smaller than unity for the finite sum to exist.

Another interesting finite sum based on the geometric series occurs for p=exp(-x)

Here we find-

$$\sum_{1}^{n} \exp(-kx) = [\exp(-x)\exp(-xn)-1]/[\exp(-x)-1]$$

For n going to infinity and x=1, this result states that-

$$\sum_{k=0}^{\infty} \exp(-n) = \frac{1}{[1-\exp(-1)]} = 1.581976..$$

Sometimes the elements of a finite series sum are easy to find but the final functional form is not so obvious. Consider, for example, -

$$T(n)=\sum_{k=0}^{n}(2^k+k)$$

Here we have-

T(0)=1

T(1)=4

T(2)=10

T(3)=21

T(4)=41

T(5)=78

T(6)=148

A first glance suggest no obvious functional form which can reproduce all the T(n). However further thought says-

$$T(n)=\sum_{k=0}^{n}(2\text{^}k) + \sum_{k=0}^{n} k = 2\text{^}(n+1)-1 + n(n+1)/2$$

This follows from using some of the earlier results. Further manipulations then yields the functional form-

T(n)=[2^(n+2)-2+n(n+1)]/2

So we can use this result to confirm all of the above values for T(n). Also we see that T(7)=283 and T(8)=547.

We have shown that many finite series may be represented by simple functional forms of n. Resemblances to both sum of the integer series and geometric series often lead to very simple summation values. A little thought often allows one to cast such finite series into simple functions of n.

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