PROPERTIES OF THE TAU AND SIGMA FUNCTION OF NUMBER THEORY

INTRODUCTION:

It is well known that any positive integer can be written as the product of primes taken to a specified integer power. Thus

$$N= (p_1\text{^}a))*(p_2\text{^}b)*(p_3\text{^}c)\dots$$

, where p_n are the ascending primes $p_1=2$, $p_2=3$, $p_3=5$ and a, b, c,.. its integer powers. For the number N=16 we have as divisors-

[1,2,4,8,16]

with the total number of divisors being called the tau function equal here to $\tau(16)=5$.The sum of the divisors is known as the sigma function and equals $\sigma(16)=31$. It is our purpose here to discuss the properties of the tau function and its link to the sigma function.

DETERMINING TAU(N):

In the above case of N=16=2^4 we see the 2^n appears four times. This is one less than $\tau(16)=5$. So to get tau(N) for a more general N we have-

$$\tau(N)=(a+1)(b+1)(c+1)\dots$$

For N=78=2^1*3^1*13^1* we get-

$$\tau(78)=(1+1)(1+1)(1+1)=8$$

As another integer take N=72000=(2^5)(3^2)(5^3). So tau(N)=(5+1)(2+1)(3+1)=72. Thus $\tau(N)$ for any positive integer N just represents the product of the exponent in the ifactor of N representation with one added. So, for instance,-

N=80,000 has ifactor(N)=2^7*5^4 so that $\tau(N)=(7+1)(4+1)=40$

When N equals a single prime p, tau will always be tau(p)=2. When N equals p^2, the value of tau becomes tau(p^2)=3. For N=p^n we have tau(p^n)=(n+1).

FINDING SIGMA((N):

To find the connection between the sigma function and the tau function we first write out the following table-

N	τ	σ
2^1	2	3
2^2	3	7
2^3	4	15
2^4	5	31
2^5	6	63
2^6	7	127
2^7	8	255

On studying this table we see at once that-

$$\sigma=2^{\tau}-1 \quad \text{for } N=2^{n}$$

Next look at the second table-

N	τ	σ
3^1	2	4
3^2	3	13
3^3	4	40
3^4	5	121
3^5	6	364
3^6	7	1093
3^7	8	3280

From it we conclude that –

$$2\sigma=3^{\tau}-1 \quad \text{for } N=3^{n}$$

Finally a third table involving N=5^n reads-

N=3^n	τ	σ
5^1	2	6
5^2	3	31
5^3	4	156
5^4	5	781
5^5	6	3906
5^6	7	19531
5^7	8	97656

From it we find-

$4\sigma=5^\tau-1$ for $N=5^n$

We can generalize the last three results as-

$$\sigma=(p^\tau-1)/(p-1) \quad \text{with } N=p^n$$

So if $N=7^3=343$ we have $\tau=4$ and $\sigma=400$.

We are now in a position to find both τ and σ for any positive integer N by first taking the ifactor of N and then working out it exponent sum to get τ and then finding the product of each exponential terms sigma to get the entire σ product. Let us demonstrate things for the number –

N=46821 whose ifactor reads (3^2)(11^2)(43^1)

This yields –

$\tau(N)=(2+1)(2+1)(1+1)=18$ and $\sigma(N)=13*133*44=76076$

Thus if we know the value of tau(N), which is easy to calculate, there is no problem finding sigma(N).

CONCLUDING REMARKS:

We have shown how both the tau and the beta functions are generated starting with the ifactor representation of any positive integer N. After carrying out the easy calculation for tau(N), we generate σ(N) by working out the sub values of σ(N) and then taking the product to get σ(N). In general we have τ<<σ and $\sigma\approx p^{\tau-1}$ as p gets large.

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