VARIATIONAL ESTIMATE FOR THE TAYLOR NUMBER AT THE ONSET OF INSTABILITY IN ROTATIONAL COUETTE FLOW

In my 1961 PhD dissertation I looked at the general problem of the effect of externally applied magnetic fields on the stability of curved viscous flows between concentric cylinders . In the non-magnetic case this reduces to the famous Couette instability problem first examined in detail by G.I.Taylor during the 1920s and later to receive considerable attention in connection with the processes involving the path to chaos in dynamical systems. Our main contribution to this problem was the observation that magnetic fields will generally hinder the critical Taylor number at which hydromagetic Couette flow first becomes unstable and that observed secondary flow cells become elongated in the field direction. In our dissertation we introduced a modified variational approach for determining the critical Taylor number T at secondary flow onset as a function of disturbance wave number 'a' and the magnetic Hartman number. The problem required finding of the eigenvalues of certain eight order ordinary differential equations subjected to a set of complicated boundary conditions . We accomplished our task by applying a variational technique to an equivalent set of lower order simultaneous equations .

Let me demonstrate this variational approach by applying it to the simpler non-magnetic case where the problem reduces to finding the lowest eigenvalues T as a function of wave number 'a' for a sixth order differential equation. The mathematical problem involves solving the equation set-

$$(4D^2 - a^2)^2 u + Ta^2 v = 0$$
$$(4D^2 - a^2) v - u = 0$$

subjected the six boundary conditions- **u(±1)=Du(±1)=v(±1)=0** with **D=d/dz.** Since one is mainly interested in finding the lowest eigenvalue Tas a function of 'a', one need not solve things exactly but rather can proceed with a variational approach employing a modified Galerkin method in which the eigenfunctions u(z) and v(z) are represented by the even expansions-

$$u(z) = (1 - z^2)^2 \sum_{n=0}^{N} c_n z^{2n} = \sum_{n=0}^{N} c_n U_n(z)$$
$$v(z) = (1 - z^2) \sum_{m=0}^{N} d_m z^{2m} = \sum_{m=0}^{N} d_m V_m(z)$$

Note that each of the trial functions $U_n(z)$ and $V_m(z)$ in these expressions satisfy the stated boundary conditions of the problem and has even symmetry in the range $-1<z<+1$. If one now multiplies the first of the above differential expressions by $U_n(z)$ and the second by $V_m(z)$ and then integrates the result over the required range of z, there results a matrix expression whose determinant must vanish for non-trivial values of the expansion coefficients c_n and d_m.

Mathematically one has-

$$F(a,T)=\begin{vmatrix} A_{1,1} & A_{1,2} & \cdots & Ta^2A_1B_1 & Ta^2A_1B_2 & \cdots \\ A_{2,1} & A_{2,2} & \cdots & Ta^2A_2B_1 & Ta^2A_2B_2 & \cdots \\ \cdots & \cdots & \cdot\cdot & \cdots & \cdots & \cdots \\ -A_1B_1 & -A_1B_2 & \cdots & B_{1,1} & B_{1,2} & \cdots \\ -A_2B_1 & -A_2B_2 & \cdots & B_{2,1} & B_{2,2} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}=0$$

where the elements in this determinant equation are given(after integration by parts)as-

$$A_{n+1,m+1}=a^4<U_nU_m>+8a^2<(DU_n)(DU_m)>+16<(D^2U_n)(D^2U_m)>$$

$$A_{n+1}B_{m+1}=<U_nV_m>$$

$$B_{n+1,m+1}=-4<(DV_n)(DV_m>-a^2<V_nV_m>$$

Here the bra-ket notation < > indicates integration over the range -1<z<1.To the lowest order(N=0) one finds-

$$T=\frac{-A_{1,1}\ B_{1,1}}{a^2(A_1B_1)^2}=[\frac{28}{27}]\frac{(504+24a^2+a^4)(10+a^2)}{a^2}$$

which already gives quite good results yielding a minimum of T=1749.9 at a=3.1165. This compares with an exact numerical value of T min =1707.76 at a=3.117. Improvements in our variational estimate are gotten by taking more terms in the trial function approximation. Although such evaluations where a time consuming task in the pre-desktop PC era of the 1960s, it is now a relatively simple matter. I recently(March 2005) carried out such an evaluation up to N=2 using the indicated trail functions. Each of the 36 elements in the resultant 6x6 matrix equation for the third approximation are expressible in closed form as Beta functions. Here are our results for T at the critical wave number of a=3.117 -

First Appoximation: **N=0 ,T(3.117)=1749.97576 ;**

Second Approximation: **N=1, T(3.117)=1708.54981;**

Third Approximation: **N=2, T(3.117)=1707.762136**

The last result is essentially identical with the exact numerical value obtained by Reid and Harris(Physics of Fluids,1,102-110(1958). Due to the variational character of our calculation method, the lowest eigenvalue found for a given wave number will always approach the exact value from above, but, as seen from our results, approaches the exact value very rapidly as more terms are used. in the trial function expansion. It is also possible to get good estimates for the velocity perturbations u(z) and v(z) for a given T and 'a' by noting that the ratios c_i/c_j and d_i/d_j of the expansion coefficients are proportional to the cofactor ratios of the appropriate elements from the first row in the above determinant equation F(a,T)=0.