TELESCOPING TERMS IN INFINITE SERIES

There are two ways to telescope terms in infinite series in order to find their total sums. The first, and more common one, retains only the first and last term in the series while eliminating all intermediate terms. The second approach condenses the first m terms of the series into a single term and then repeats the procedure for the next terms. Eventually this reduces the sum evaluation by a factor of m. It is the purpose of this article to discuss in more detail these two telescoping approaches.

e=EXP(1):

We begin by looking at the series for the constant e. It reads-

$$e=\sum_{k=0}^{\infty} 1/n! =1+/1!+1/2!+1/3!!+$$

Taking two terms at a time we get the series-

$$e=(1+1)+(1/2!+1/3!)+(1/4!+1/5!)+\ldots=2+2/3+1/20+\ldots$$

More generally we have for the first two term m packet –

$$(1/n!)+1/(n+1)!=(n+2)/(n+1)!$$

The second and higher packets follow by replacing n by 2n, to get the telescoped series–

$$e=\sum_{k=0}^{\infty} \frac{2(n+1)}{(2n+1)!}=2/1!+4/3!+6/5!+8/7!+$$

This interesting series for e converges faster by a factor of two.

Going on with a four term packet, we find–

1/n!+1/(n+1)!+1/(n+2)!+1/(n+3)!=(n^3+7n^2+17n+16)/(n+3)!

On replacing n by 4n we get the new series-

$$e=\sum_{n=0}^{\infty} \frac{(64n^3+112n^2+68n+16)}{(4n+3)!} \quad .$$

Here each step contains the sum of four original elements. Taking the sum up to k=5 already yields an approximation for e good up to 24 digits.

$S=\sum_{n=0}^{\infty} \frac{1}{(n^2+4n+3)}$:

We first rewrite this series as-

$$S=\left(\frac{1}{2}\right) \sum_{n=0}^{n=\infty} \frac{1}{(n+1} - \frac{1}{n+3} \quad .$$

Expanding we get-

S=(1/2){1/1-1/3+1/2-1/4+1/3-1/5+1/4-1/6+…}

This time all terms after ½ vanish . Thus we are left with-

$$S=\sum_{n=0}^{\infty}\frac{1}{(n^2+4n+3}=(\frac{1}{2})\sum_{n=0}^{n=\infty}[\frac{1}{(n+1}-\frac{1}{n+3}] = [1+1/2)]/2=3/4$$

Here the first sum yields a very slowly converging series-

1/3+1/8+1/15+1/24+1/35+…=3/4

, whose value was made possible by telescoping. Note that making packets from this series would not be very helpful. Also the terms in the second sum form are close to a harmonic series which is known to diverge. So here we have the interesting result that the difference of two infinities equals something finite

$\boldsymbol{T = \sum_{n=0}^{\infty}\frac{n}{n^4+1}}$:

We next examine the infinite series $\cancel{T =}\sum_{n=0}^{\infty}\frac{n}{n^4+1}$ which converges to the finite value 0.694173022… The standard ratio test of calculus shows that this series must be convergent although one does not know the exact value until the sum has been evaluated term by term. The series reads-

T=1/2+2/17+3/82+4/257+5/626+6/1297+

The six terms shown sum to 0.68240 and indicate things are converging to 0.69417. We can speed up the convergence by telescoping two terms at one time by replacing- n by 2n in-

n/(n^4+1)+(n+1)/((n+1)^4+1)

This gives us the alternate two series form-

$$T=\sum_{n=0}^{\infty}\frac{2n}{(2n)^4+1)}+\sum_{n=0}^{\infty}\frac{(2n+1)}{((2n+1)^4+1)}$$

Expanding yields-

T=(0+2/17+4/257+6/1297+…)+(1/2+3/82+5/626+7/2402+…)

In this case further telescoping would not be very helpful since the sum elements become cumbersome.

$\boldsymbol{V=\sum_{n=0}^{\infty}\frac{(-1)^{\wedge}n}{(n^2+1)}}$:

The last of the sums we want to look at is the alternating sign sum shown. It sums to 0.6360145

and has the series representation-

V=1-1/2+1/5-1/10+1/17-1/26+1/37-1/50+It leds to the two term telescoped packet-

$$V=\sum_{n=0}^{\infty}\frac{(4n+1)}{(4n^2+1)(4n^2+4n+2)} = 1/2+1/10+9/442+$$

Note that this time all elements of the series are positive meaning we are approaching the final sum from below.

We have shown that infinite series can be summed more rapidly by use of the telescoping method. The method works well as long as the telescoped terms don't become too large. All of the above examples discussed show the advantage of telescoping.

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