TRICKS WHICH CAN BE USED TO SOLVE CERTAIN DEFINITE INTEGRALS

You probably remember from your calculus class that there exists the identity-

$$F(b)-F(a)=\int_a^b f(x)dx \quad \text{where } dF(x)/dx=f(x)$$

Related to this fundamental theorem one also has the Leibnitz-Feynman rule of differentiation under the integral sign. It reads-

$$d/dz[\int_{a(z)}^{b(z)} f(x,z)dx] = \int_{a(z)}^{b(z)} \frac{df(x,z)}{dz} dx + f(b(z),z)db(z)/dz - f(a(z),z)\, da(z)/dz$$

Use of these two formulas and certain exponential identities such as-

exp(ix)=cos(x)+isin(x) , cos(x)^2+sin(x)^2=0 and i^n=i^(n-4)

allows one to evaluate many types of definite integrals. It is our purpose here to show that often very complicated looking definite integrals produce very simple results.

Let us begin with the integral-

$$I(a,b)= \int_{x=0}^{\infty} \sin(ax)\exp(-bx)\, dx \text{ , } where\ a\ and\ b\ are\ specified\ positive\ real\ numbers$$

To solve it we note that Im(exp(iax))=sin(ax) . So I(a,b)=Im $\int_{x=0}^{\infty} \exp x(-b+ia))\, dx$. On integrating and putting in the specified limits, we find-

I(a,b)=Im(1/(b-ia)=a/(b^2+a^2)

Replacing sin(ax) by cos(ax) will change the result to b/(b^2+a^2).

Next we look at –

$$J(n,a)= \int_{x=0}^{\infty} x^n \exp(-ax)\, d\, x$$

The fastest way to evaluate this definite integral is to note that this is the Lapalce transform of x^n with s replaced with a. But lapalce(x^n)=n!/s^(n+1). Hence we have-

J(n,a)=n!/a^(n+1

Setting a=1 and n to n-1, we get the important identity that the Gamma Function equals-

$$\Gamma(n)=\int_{x=0}^{\infty} x^{(n-1)}\, exp(-x)\, dx = (n-1)!$$

On setting n=1/2 we have the identity-

$$sqrt(\pi)=\int_{n=0}^{\infty} \exp(-x)\, /sqrt(x)dx$$

Consider next the identity-

M(a,b)=$\int_{x=0}^{\infty} \frac{\sin(ax)\exp(-bx)}{x} dx$= arctan(a/b)

How is this derived?. One way is to go under the integral sign(Leibnitz-Feynman Technique) and differentiate M with respect to a.This gets rid of the x in the denominator to yield-

dM/da= $\int_{x=0}^{\infty} \cos(ax)\exp(-bx)\, x = b/(b^2 + a^2)$

Solving for M we get-

M=$\int b/(b^2 + a^2)$da=arctan(a/b)

after setting the integration constant to zero. Note that on setting a=1 and b=0, we get the interesting result that-

$\int_{x=0}^{\infty} (\sin(x)\,/x)dx = \arctan(\infty) = \pi/2.$

Another integral of zero to infinite limit is-

N(a)=$\int_{x=0}^{\infty} \exp(-ax^2)\, dx$ with the constant 'a' positive and real

Letting u=(1/(2sqrt(aax^2 we find-

N(a)=(1/(2sqrt(a))$\int_{x=0}^{\infty} \exp(u)\,/sqrt(u)du\, x = \Gamma(0.5)/2sqrt(a)$

On setting a=1 we get the important result-

N(1)=$\int_{n=0}^{\infty} \exp(-x^2)\, dx$ =sqrt(π)/2

In all of the above cases we had x extend from 0 to infinity . There are an infinite number of definite integrals where the range of x is much smaller. Take for example-

T= $\int_{x=0}^{1} \frac{\ln(x)}{1+x} dx$

To solve this definite integral we can expand 1/1+x) in a geometric series to get-

T=$\sum_{n=0}^{\infty}(-1)^{\wedge}n \int_{x=0}^{1}(x^{\wedge}n)\ln(x)dx$

But we know the integral of x^n ln(x) from x= 0 to 1 is –

$\int_{x=0}^{1} x^n \ln(x)\, dx = -1/(n+1)^{\wedge}2$

So we have the infinite sum-

T=$\sum_{n=0}^{\infty}(-1)^{n+1}/(n+1)^{\wedge}2$=-1/1+1/4-1/9+1/16-…

This starts to look a lot like the Euler result-

π/6=1+1/4+1/9+1/16+…

Indeed we find T+π/6=(1/2)(π^2/6). This means that-

$$\int_{x=0}^{1} \frac{\ln(x)}{1+x} dx = -\frac{\pi}{12} = 0.261799\ \ldots$$

As another integral with finite range, consider-

$$V=\int_{n=0}^{1} \frac{1-x}{(1+x)} dx$$

This is easy to integrate by the variable substitution u=1+x. It yields-

$$V=\int_{x=1}^{2} \frac{2-u}{u} du=-1+2\ln(2)$$

As a bit more complicated finite integral consider-

$$W(a)=\int_{x=0}^{2\pi} dx/(1+\operatorname{acos}(x)) = 2\pi/sqrt(1-a^2)$$

Here we make use of the identity-

Cos(2z)=2cos(z)^2-1 with x=2z

A little manipulation then produces-

$$W(a)=\int_{z=0}^{\pi} 2\, dz/((1-a)+2\,\mathrm{a}\cos(z)^2)$$

This integral can be evaluated exactly for 0<a<1, yielding-

W(1/2)=4π/sqrt(3), W(1/4)=8π/sqrt(15), and W(1/8)=16π/63

From these results a generalization yields-

W(1/n)=2nπ/sqrt(n^2-1) or the equivalent $W(a)=\frac{2\pi}{sqrt(1-a^2)}\ with\ 0 < a < 1.$

As the above calculations have shown, there are many different ways to evaluate definite integrals. The most powerful approach is to use variable substitutions followed by integration under the integral sign, followed by use of partial fractions, and finally using series expansions. If all of these fail, it is always possible to integrate a definite integral to any desired degree of accuracy numerically provided one carefully treats any singular points lying in the range of integration.

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