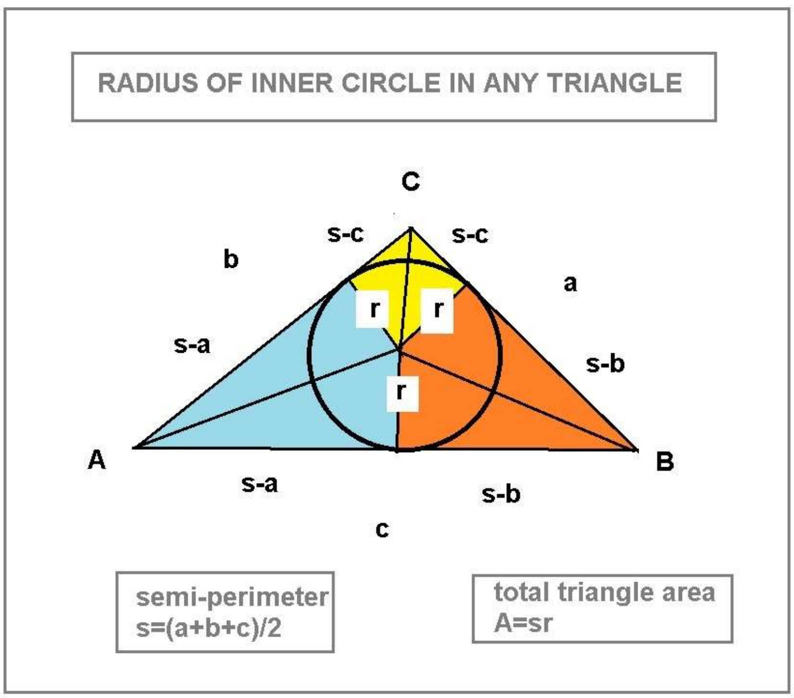
INSCRIBED AND CIRCUMSCRIBED CIRCLES FOR ANY TRIANGLE

It is well known that there are two unique circles centered at the centroid of any <u>equilateral</u> triangle of side-length L. These have radii r=L/(2sqrt(3) and R=L/sqrt(3) .The later circumscribes the triangle while the former lies inside the triangle. When dealing with scalene triangles the center of the smaller circle remains at the centroid but the larger circle will be located at a different point. Both circles will have radii more complicated than those given for the equilateral triangle. It is the purpose of this note to find these two radii for any triangle and the new center point for the larger circle.

CIRCLE INSIDE A TRIANGLE:

Here we begin by drawing a triangle of sides a,b, and c with vertexes A, B, and C as shown-



Next we bisect the angles A, B, and C with straight lines heading toward a common center which is the location of the centroid of the triangle. At this centroid a circle of radius r which just touches the triangle boundaries can be drawn. This represents the desired inscribed circle in the triangle. Some simple arithmetic now gives the entire triangle area as –

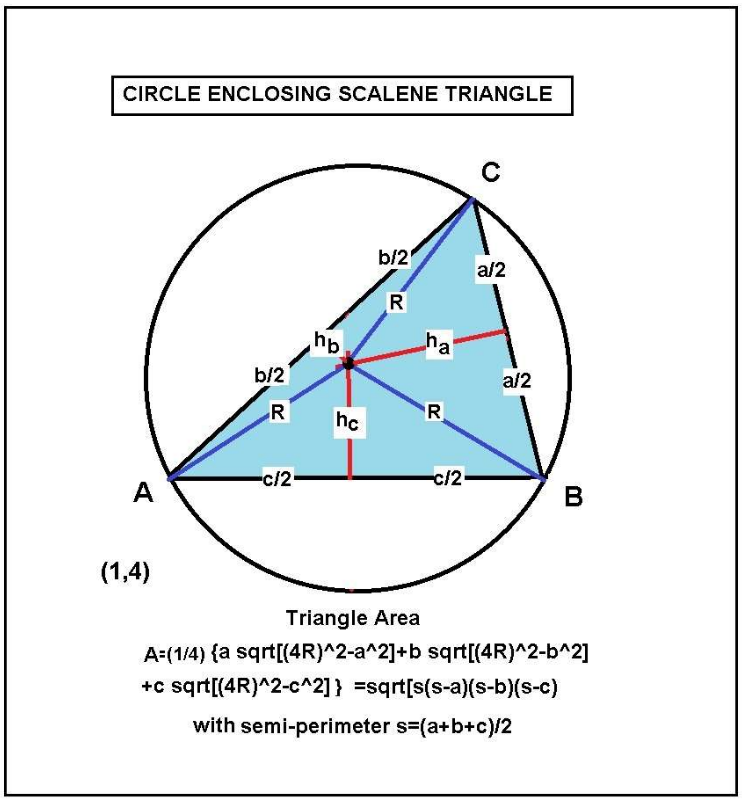
$$A=r[(s-a)+(s-b)+(s-c)]=r(a+b+c)/2=rs$$

Using the Heron Formula we can rewrite this last result to yield the exact radius of the inner circle to be-

$$r=\text{sqrt}\{[(s-a)(s-b)(s-c)]/s\}$$

CIRCLE CIRCUMSCRIBING A TRIANGLE:

The next circle we examine is one which has a larger radius R and circumscribes any triangle of vertexes A, B, ad C and sides a, b, and c. We have the following picture if we place the largest side c horizontally with its halfway point at Cartesian coordinate x=y=0-



We next draw three redlines perpendicular to the center points of the three triangle sides. These meet at a common center point which defines the center of a new circle. Note that the circle center lies at coordinates –

$$[x,y]=[0,h_c]=[0,0]$$

To find the exact value of R, we use two different versions of the triangle area. The second of these is the Hero Formula and the first is obtained by adding up the sub-triangles in the above figure using the Pythagorean Theorem. We find-

A=(1/2){a sqrt[R^2-(a/2)^2]+b sqrt[(R^2-(b/2)^2]+ c sqrt(R^2-(c/2)^2]}
=sqrt[s(s-a)(s-b)(s-c)]

Equating the two areas we have an implicit formula for finding R.

EVALUATION OF r AND R FOR SPECIFIC CASES:

We can easily evaluate the above formula for r by hand once a, b and c are known. Also the last formula for A allows one to quickly evaluate R using a simple one line computer program. Again all that is required is that one knows the values a, b, and c.

Let us begin with a=6, b=8 and c=10. Here the semi-primeter is s=12 and by the Heron Formula we have A=24. Thus r=A/s=2. Next we solve-

48=6sqrt(R^2-3^2)+8sqrt(R^2-4^2)+10sqrt(R^2-5^2)

to get R=5.This last circle is centered at [x,y]=[0,0]which is the middle of line c=10. This point differs (unlike for equilateral triangles where both circle origins coincide) from the centroid location of the small circle.

Consider next the case a=4, b=5 and c=6. Here s=15/2 and A=sqrt[(15/2)(7/2)(5/2)(3/2)]=15sqrt(7)/4=9.921567.. . So the small circle radius becomes r=sqrt(7)/2=1.32287.. and the large circle radius is R=3.02371…by solving-

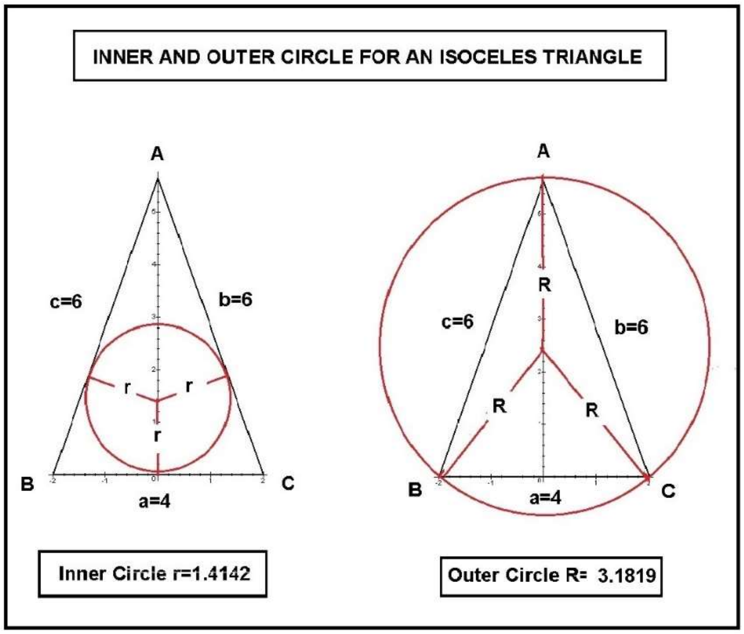
15sqrt(7)/2 =4sqrt(R^2-4)+5sqrt(R^2-(5/2)^2)+6qrt(R^-9)

Note here that the ratio R/r=3.02371/1.32287=2.2857. This lies fairly close to the two to one ratio for an equilateral triangle meaning that the last triangle does not depart much in shape from an equilateral triangle.

As one last example for the inner and outer circles to a triangle, consider the isosceles triangle a=4, b=6, and c=6. Here the semi-perimeter is s=(4+6+6)/2=8 and the area is A=sqrt[8(4)(2)(2)]=8sqrt(2). The radius of the inner circle becomes r=8sqrt(2)/8=sqrt(2). Also we have-

4sqrt(2)=sqrt(R^2-2^2)+3sqrt(R^2-3^2)

This solves as R=3.18198 . The radius ratio equals R/r=2.2503. The following gives the two circles for the triangle-



You will note that both circles have their center fall on the same vertical symmetry line, but their origins fall at different points along this line.

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