FORMULAS FOR SIGMA, TAU, AND NUMBER FRACTION FOR p^n

It is well known that any positive integer may be represented by a finite product of primes taken to specified powers. That is-

$$N=\prod_{k=1}^{k=m} p_k \hat{\ } n_k$$

Clearly this equality will be satisfied if we know the values of all the basic building blocks p^n. These blocks in turn have their own unique values for the tau, sigma and number fraction encountered in number theory. It is our purpose here to find explicit values for these quantities expressed exclusively in terms of just primes p and integer exponents n.

We start with the definition of the sigma function for the special case of p^n. Using the geometric series, It reads-

$$\sigma(p\hat{\ }n)=1+p+p\hat{\ }2+....p\hat{\ }n= 1/(1-p)-\prod_{k=n+1}^{\infty} p\hat{\ }k$$

Letting j=k-(n+1), the product term goes as $p\hat{\ }(n+1)\prod_{j=0}^{\infty} p\hat{\ }j$.

So we are left with-

$$\sigma(p\hat{\ }n)= [p\hat{\ }(n+1)-1]/(p-1) \quad .$$

This result works for all primes p when integer n equals one or greater.

Next we look at the tau function for p^n. This function represents the number of divisors the function p^n has. This number here will be n+1. Hence tau takes on the very simple form-

$$\tau(p\hat{\ }n)=n+1$$

It depends only on the exponent n.

As a third number theory function we look at the number fraction defined for p^n as-

$$f(p\hat{\ }n)=[\sigma(p\hat{\ }n)-p\hat{\ }n-1]/p\hat{\ }n= [1-p\hat{\ }(1-n)]/(p-1)$$

So, as predicted, the numbers σ, τ , and f for p^n depend only on p and n.

Let us work out some specific cases. Take first p=13 with n=4. Here we have-

$$p^n=13^4=28561$$

This produces-

$\sigma(13^4)=(13^5-1)/(13-1)=30941$, $\tau(13^4)=n+1=5$, and

$$f(13^4)=(1-1/13^3)/(13-1)=0.0832954\ldots$$

Notice here that σ lies slightly above p^n and f equals about $1/12=0.083333\ldots$

Consider next the much larger prime $p=48611$ taken to the 15th power. It has-

$\tau(p^n)=16$,

$\sigma(p^n)=$
$1999987414928349859794080743315287558028421021418051711023807137290921 6$ and

$f(p^n)=.000020571898786257971611\ldots$

As also seen in the previous example, the value of-

$p^n=$
$1999946272236059465647492644310055917297762766681028855050652423190465 1$ lies close to $\sigma(p^n)$ and $1/p=0.00002057147559$ lies close to $f(p^n)$.

Note also the relation between τ and σ given by –

$$\sigma(p^n)=(p^\tau-1)/(p-1)$$

So knowing $\tau(p^n)$ means we at once have the value of $\sigma(p^n)$.

Consider next the double p^n number-

$$N=(7^3)(5^4)=214375$$

It has its σ number given by-

$$\sigma(N)=(7^4-1)/(7-1)*(5^5-1)/(5-1))=400*781=312400 \quad .$$

If one deals with semi-primes $N=pq$, we can find the primes p and q by noting here that $n=1$. This produces the equality-

$$\sigma(N)=(p+1)(q+1)=N+(p+q)+1$$

Upon eliminating either q or p, we get the quadratic

$$[p,q]^2-[p,q](\sigma(N)-N-1)+N=0-$$

So, if σ(N) is known, p and q will be given. Take the case of N=77 where σ(N)=96. It produces –

$$[p,q]^2-[p,q]*18 +77=0$$

The factors are thus p=7 and q=11. This last result can be generalized for any N provided σ(N) can be generated in a few seconds. On my home computer (ThinkPad with use of Maple) I can readily find σ(N) out to about 40 digit length in several seconds. The secret for breaking semi-primes of one hundred or larger digit length (as used in modern day cryptography) will require some new technique for quickly finding σ(N) at still larger N.

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January 4, 2024
Gainesville, Florida