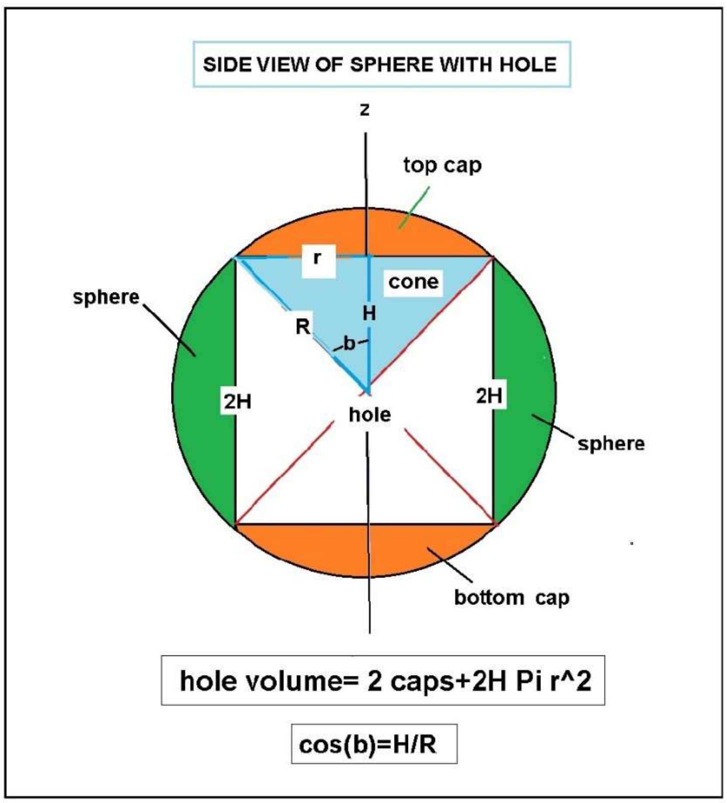
VOID FRACTION PRODUCED BY DRILLING A CYLINDRICAL HOLE THROUGH THE CENTER OF A SPHERE

Consider a solid sphere of radius R through which is drilled a cylindrical hole of radius r. It is assumed that r<R so that the void fraction, equal to the ratio of hole volume to sphere volume, lies in the range 0<ϵ<1. It is our purpose here to show that the resultant void fraction equals-

$$\epsilon=[1-\sqrt{1-x^2}\ +x^2\sqrt{1-x^2}\ ]$$

using a combination of both spherical and cylindrical geometric calculations.

To prove this result, we start with the following sketch-



This shows the side-view of a sphere of radius R with a vertical cylindrical hole of diameter 2r drilled into it. Each of the two spherical caps shown in orange have the volume-

$$V_{cap}= [\,2\pi \int_{r=0}^{R} r^2 dr \int_{\theta=0}^{b} \sin(\theta)\, d\theta] - \pi r^2 H/3 \quad \text{with}\ \cos(b)=H/R=\sqrt{1-x^2}\ \ and\ x=r/R$$

On integrating this volume reads-

$$V_{cap}=(\pi\frac{R^3}{3})[2-\sqrt{1-x^2}\,(2+x^2)\,]$$

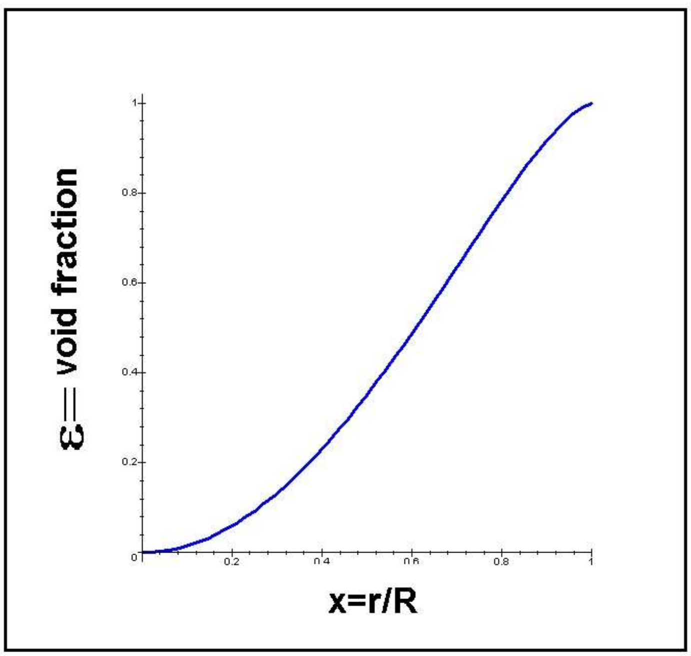
Note here that the term $\pi r^2H/3$ in the above equation represents the cone shown in blue. Next we look at the entire volume of the hole. It equals-

$$V_{hole}=2V_{cap}+2\pi r^2H=[4\pi R^3/3]\{1-\sqrt{1-x^2}+x^2\sqrt{1-x^2}\}$$

You will recognize that $V_{sphere}=4\pi R^3/3$ is just the volume of the original sphere of radius R. This leaves us with the definition of the void fraction as-

$$\varepsilon=(V_{hole}/V_{sphere})=\{1-\sqrt{1-x^2}+x^2\sqrt{1-x^2}\}$$

Recall that x=r/R, so that $\varepsilon=0$ when x vanishes and $\varepsilon=1$ for x=1. At x=0.5 we find $\varepsilon=0.6083087$. Here is a plot of hole to sphere radius versus epsilon-



For small holes such as used in stringng together a pearl necklace, the value of epsilon equals-

$$\varepsilon\sim(\frac{3}{2})x^2 \quad \text{with} \quad x<0.1$$

We can also calculate the surface area of the top of one of the orage caps. Using spherical coordinates, we find the area to be-

$$S_{cap}=R^2\int_{\theta=0}^{b}\sin(\theta)\,d\theta\int_{\varphi=0}^{2\pi}d\varphi=2\pi R^2[1-sqrt(1-x^2\,)]$$

If x=1, the cap has an area of half the sphere surface.

We can also go on and find the circumference of a great circle for a sphere of radius R. Using spherical coordinates, we have that the circumference of a circle of radius r, circumscribing the above described hole, is –

$$C=2\pi r= 2\pi Rx \quad \text{with} \quad x=r/R$$

The circumference of the sphere's equator is then is found by setting x=1.This circumference equals the well known result $C=2\pi R$. When this circle is made to pass through two points A and B on the sphere's surface, one has what is known as the great circle route. It is the shortest distance, although on a Merkator (Flemish cartographer 1512-1594) projection, it appears longer. Another way to look at such a geodesic path through points A and B is to say that it represents the intersection of a plane passing through the sphere center and the sphere surface.

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My 86th Birthday