VALUES OF THE RIEMANN ZETA FUNCTION

One of the better known mathematical functions is the Riemann Zeta Function defined as-

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \ldots = \prod_{n=1}^{\infty} \frac{1}{n^s}$$ provided that real \(s > 1\)

Its values for other \(s = \sigma + \imath \tau\) are determined by analytic continuation. The function can be multiplied by \((1-1/2^s)\) to yield-

$$\left(1 - \frac{1}{2^s}\right) \zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} + \frac{1}{11^s} + \frac{1}{13^s} + \frac{1}{15^s} + \ldots$$

, remembering that we keep the real part of \(s\) greater than one. Also we have-

$$\left(1 - \frac{1}{2^s}\right) \left(1 - \frac{1}{3^s}\right) \zeta(s) = 1 + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{11^s} + \frac{1}{13^s} + \frac{1}{17^s} + \frac{1}{19^s} + \frac{1}{23^s} + \frac{1}{25^s} + \ldots$$

Continuing one reaches -

$$\prod_{n=1}^{\infty} \left(1 - \frac{1}{p(n)^s}\right) \zeta(s) = 1$$

, where \(p(n)\) is the \(n\)th prime. Rewriting this last result yields the remarkable formula-

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{n=1}^{\infty} 1/[1 - \frac{1}{p(n)^s}]$$

, first discovered by Leonard Euler some 250 years ago. It was one of the first functions found expressible both as an infinite sum and an infinite product. At \(s = 2\) it reads-

$$\zeta(2) = \frac{4 \cdot 9 \cdot 25 \cdot 49 \cdot 121 \cdot 169}{3 \cdot 8 \cdot 24 \cdot 48 \cdot 120 \cdot 168} \ldots = 1.644934067\ldots$$

Next looking more at the expression \((1-1/2^s)\zeta(s)\), we find-

$$\zeta(s) = \frac{2^s}{(2^s - 1)} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s}$$
With all terms in the sum being reciprocal of odd integers. If one takes \( s=2 \), we re-obtain the earlier result:

\[
\zeta(2) = \frac{4}{3} \left\{ 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81} + \frac{1}{121} + \ldots \right\} = 1.6449340668482264\ldots
\]

It was first shown by Euler, by another means, that this result is equivalent to \( \zeta(2) = \frac{\pi^2}{6} \).

It will be our purpose here to use some of the above formulas to find additional values of the Zeta Function \( \zeta(s) \) for different \( s \) including complex ones.

We begin by re-writing \( \zeta(2) \) as-

\[
\frac{\pi^2}{6} \cdot \frac{3}{4} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}
\]

Taking the difference of \( \zeta(2) \) with this produces-

\[
\frac{\pi^2}{24} = \sum_{n=1}^{\infty} \left\{ \frac{(3n-1)(n-1)}{n^2(2n-1)^2} \right\} = \frac{5(1)}{4(9)} + \frac{8(2)}{9(25)} + \frac{11(3)}{16(49)} + \ldots = 0.4112335167\ldots
\]

Another identity which follows from the above formulas is-

\[
\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{-2n^2 + 4n - 1}{n^2(2n-1)^2}
\]

We continue by looking at \( \zeta(3) \) and \( \zeta(4) \). We can here use the above series for \( \zeta(s) \) to write-

\[
\zeta(3) = \frac{8}{7} \left\{ 1 + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \frac{1}{9^3} + \frac{1}{11^3} + \ldots \right\} = 1.20205690915095942854\ldots
\]

and-

\[
\zeta(4) = \frac{16}{15} \left\{ 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} + \frac{1}{11^4} + \ldots \right\} = 1.0823232337111381916\ldots = \frac{\pi^4}{90}
\]

Continuing on we get-
\[
\zeta(5) = \frac{32}{31} \left\{ 1 + \frac{1}{3^5} + \frac{1}{5^5} + \frac{1}{7^5} + \frac{1}{9^5} + \frac{1}{11^5} + \ldots \right\} = 1.0369277551433699263\ldots
\]

and-

\[
\zeta(6) = \frac{64}{63} \left\{ 1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{9^6} + \frac{1}{11^6} + \ldots \right\} = 1.0173430619844491398\ldots = \frac{\pi^6}{945}
\]

We note in all the above examples that \( \zeta(2n) \) has values given as \( \frac{\pi^{2n}}{\text{integer}} \) as long as \( n \) remains 12 or less. The odd function \( \zeta(2n+1) \) does not obey such a law. A few additional even Zeta Functions yield the values of \( \zeta(8) = \frac{\pi^8}{9450}, \zeta(10) = \frac{\pi^{10}}{93555}, \) and \( \zeta(12) = \frac{\pi^{12}}{924041.7887} \). The last no longer has an integer denominator. A plot of \( \zeta(s) \) versus \( s \) follows-

Note the singularity at \( s=1 \) corresponding to the harmonic series. When \( s \) gets large we have the approximation-

\[
\zeta(8) \approx \left\{ \frac{2^8}{2^8 - 1} \right\} \left\{ \frac{3^8}{3^8 - 1} \right\} = 1.004074\ldots
\]

This result compares with the exact result \( \zeta(8) = \frac{\pi^8}{9450} = 1.004077.. \)
We continue on and look at the Zeta Function when $s=\sigma+i\tau$ is complex. Writing out the Zeta Function for $s=\sigma+i\tau$, we get -

$$\zeta(\sigma + i\tau) = \sum_{n=1}^{\infty} \frac{\cos(\tau \ln(n)) - i \sin(\tau \ln(n))}{n^{\sigma}}$$

Next choosing $s=1+i$ we find-

$$\zeta(1 + i) = \sum_{n=1}^{\infty} \frac{\cos(\ln(n)) - i \sin(\ln(n))}{n} = 0.5821580598 - i0.92684856435$$

This checks with our MAPLE computer program for the same $s$. Trying next the complex form $s=\sigma+i\tau=1/2+i2$, we find our sum to equal-

$$\zeta(1/2+2i)=0.4405456503-i0.31164633845$$

Note this time the real part of $s$ was less than one yet still the series representation gave the correct value for the Zeta Function. The line $s=1/2+i\tau$ is of historical interest because, as Bernhard Riemann first conjectured, it is the only line in the $s=\sigma+i\tau$ plane (with the exception of $s$ equal to negative even integers) where zeros exist. Here is a contour graph of the first three zeros along the $\sigma=1/2$ axis.
Notice all the circles graphed fall along the \( \tau = 1/2 \) line. We have found no other zeros except along this line and the negative even values \( s = -2n \). A formal proof will follow by looking at the location of all small circle contours in the s plane.

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