HOW DO THE INTEGER N, THE SIGMA FUNCTION $\sigma(N)$, AND THE NUMBER FRACTIN $f(N)$ RELATE TO EACH OTHER?

Over the last decade we have been examining various facets of number theory. In this endeavor we have come up with a new interesting point function defined as-

$$ f(N) = \frac{[\sigma(N) - N - 1]}{N} $$

We refer to it as the Number fraction. Here $\sigma(N)$ is the standard sigma function also known as the divisor function. It equals the sum of all of the devisors of N. Thus for $N=604$ we have –

$$ f(N)=\frac{459}{46}=0.7599337\ldots \text{ and } \sigma(N)=1064 $$

The big advantage of the number fraction $f(N)$ over $\sigma(N)$ is that it is identically zero for any prime number and that its values are greater than unity for super-composites such as $N=288$ where $f(288)=(265/144)=1.840277\ldots$. It is the purpose of this article is to discuss the various identities relating $f(N)$, $\sigma(N)$, and $N$ to each other and to show quick ways to find $f(N)$ and $\sigma(N)$ for any $N$.

We begin by noting the following when $N$ is a prime $p$-

$$ f(p) = 0 $$
$$ f(p^2) = \frac{1}{p} $$
$$ f(p^3) = \frac{1 + p}{p^2} $$
$$ f(p^4) = \frac{1 + p + p^2}{p^3} $$

Generalizing these results we find-

$$ f(p^n) = \sum_{k=1}^{n-2} p^k \frac{1 - p^{1-n}}{p^{n-1}} $$

Also noting that –

$$ \sigma(p^n) = 1 + p^n + p^n f(p^n) $$

we get-
\[ \sigma(p^n) = \frac{p^{n+1} - 1}{p - 1} \]

The ratio between \( f(p^n) \) and \( \sigma(p^n) \) yields the rational fraction

\[ R = \frac{\sigma(p^n)}{f(p^n)} = \frac{(p^{n+1} - 1)}{(p^{2n} - p^{-1})} \]

At \( p=3 \) and \( n=2 \) we find \( R=1/39 \). Note also that \( \sigma(p)=1+p \) for all primes.

Although we have found closed form solutions for the functions \( f(N) \) and \( \sigma(N) \) when \( N \) is an integer power of a prime \( p \), we have not yet extended the discussion to finding the value of these point functions when \( N \) is a composite. To do so we recall that any \( N \) maybe expanded in the form of the product of primes taken to specified powers. Thus for example-

\[ N= 418=2 \cdot 11 \cdot 19 \quad \text{and} \quad 3675=3 \cdot 5^2 \cdot 7^2 \]

To work out \( \sigma(N) \) for these two cases we can take the sigma of each of the prime components and multiply things together. We thus have-

\[ \sigma(418) = \sigma(2)\sigma(11)\sigma(19) = 3 \cdot 12 \cdot 20 = 720 \]

and-

\[ \sigma(3675) = \sigma(3)\sigma(5^2)\sigma(7^2) = 4 \cdot 31 \cdot 57 = 7068 \]

Note that here we also used \( \sigma(p^2)=1+p+p^2 \). To convert to \( f(N) \) we just write-

\[ f(N) = \frac{[\sigma(N) - N - 1]}{N} \]

Hence \( f(418)=0.7200956938\ldots \) and \( f(3675)=0.9229931973\ldots \)

It is now clear that one can determine the value of \( f(N) \) and \( \sigma(N) \) for any positive integer \( N \) be it prime or a composite. All that is necessary is to ifactor \( N \) into its products of primes taken to appropriate powers and then apply the mathematical operations to the components. One has-

\[ f(N) = \left( \frac{1}{N} \right) \{-(N + 1) + \prod_{k=1}^{b} \left[ \frac{(p_k^{a_k+1} - 1)}{(p_k - 1)} \right] \} \]

Let us demonstrate things for the eleven digit long composite number-
\[ N = 18443515310 = (2)(5)(11)(23)(61)(127)(941) \]

Here –
\[ \sigma(N) = 3 \cdot 6 \cdot 12 \cdot 24 \cdot 62 \cdot 128 \cdot 942 = 38754091008 \]

and-
\[ f(N) = 1.101231265… \]

To get into the higher super-composite range consider the composite number-
\[ N = 2^{20} \cdot 3^2 \cdot 5^3 \cdot 7 = 487599243264000 \]

Here we find-
\[ \sigma(N) = \sigma(2^{21} - 1)\sigma((3^{13} - 1)/2)\sigma((5^4 - 1)/4)\sigma(7) = \]
\[ (2097151)(797161)(156)(8) = 2086365201412128 \]

and-
\[ f(N) = 3.27885… \]

Note that the value of \( f(N) \) lies considerably above unity meaning we are dealing with a super-composite. Such numbers stand out clearly when plotting \( f(N) \) versus \( N \) as shown in the following graph-
To complete our discussion we look at the factoring of the semi-prime $N=pq$. Here we can write-

$$\sigma(N)=\sigma(p)\sigma(q)=(p+1)(q+1)=N+(p+q)+1$$

so that-

$$p+q=N\sigma(N)$$

Solving for $p$ and $q$ then produces-

$$[p, q] = \frac{N\sigma(N)}{2} \mp \sqrt{\left(\frac{N\sigma(N)}{2}\right)^2 - N}$$

with-

$$N\sigma(N)=\sigma(N)-N-1$$

Thus if $\sigma(N)$ or $\sigma(N)$ is known, the semi-prime $N=pq$ is factored into its two prime components. Most advanced computer programs such as Maple give the values of $\sigma(N)$ for $Ns$ out to 40 digit length. Thus the semi-prime-

$$N=K38=79259796022025569219580682646178858411$$

has-

$$\frac{(\sigma(N)-N-1)}{2}=N\sigma(N)/2=79259796023813883641257066123781216832$$

and the number factors into-

$$N=(44320951093)(1788314421676383433281407327)$$

all within a split second.

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